APPENDIX

FUNCTIONAL ANALYSIS

BELA SZ.-NAGY

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TO

FRIGYES RIESZ and BÉLA SZ.-NAGY

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EXTENSIONS OF LINEAR TRANSFORMATIONS IN HILBERT SPACE WHICH EXTEND BEYOND THIS SPACE

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Extensions of Linear Transformations in Hilbert Space Which Extend Beyond This Space

By BÉLA SZ.-NAGY

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LEO F. BORON

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APPENDIX

EXTENSIONS OF LINEAR TRANSFORMATIONS IN HILBERT SPACE WHICH EXTEND BEYOND THIS SPACE

Dedicated to Frigyes Riesz
on the occasion of his seventy-fifth birthday
January 22, 1955

1. Introduction

The essential structure of normal, in particular of self-adjoint and unitary, transformations in Hilbert space is known, thanks to the spectral decomposition theorem for these transformations. Much less is known about the structure of non-normal transformations because of the lack of a satisfactory generalization to Hilbert space of either the Jordan canonical form for finite matrices or of the theory of elementary divisors. This is why it is important to find relations between normal and non-normal transformations, which will enable one to reduce certain problems dealing with general linear transformations to the more workable particular case of normal transformations.

The simplest relations of this type are

$$T = A + iB,$$

where the bounded linear transformation T in the Hilbert space $\mathfrak{H}$ is represented by the two self-adjoint transformations

$$A = \operatorname{Re} T = \frac{1}{2}(T + T^*), \quad B = \operatorname{Im} T = \frac{1}{2i}(T - T^*),$$

and

$$T = VR$$

where the bounded linear transformation T in the Hilbert space $\mathfrak{H}$ is represented by the positive self-adjoint transformation

$$R = (T^* T)^{\frac{1}{2}}$$

and V is a partially isometric transformation (which, in certain cases, can be chosen to be unitary, in particular if T is normal or if T is a one-to-one transformation of the space $\mathfrak{H}$ onto itself).[1] The applicability of these relations is

[1] See Sec. 110.

restricted by the fact that neither A and B nor V and R are in general permutable, and there is no simple relation among the corresponding representations of the iterated transformations $T, T^2, \ldots$.

In the sequel, we shall deal with other relations which are connected with extensions of a given transformation. But, contrary to what we usually do, we shall also allow extensions which *extend beyond the given space.*

So, by an *extension* of a linear transformation T of Hilbert space $\mathfrak{H}$, we shall understand a linear transformation $\boldsymbol{T}$ in a Hilbert space $\mathbf{H}$ which contains $\mathfrak{H}$ as a (not necessarily proper) subspace, such that $\mathfrak{D}_{\boldsymbol{T}} \supseteqq \mathfrak{D}_T$ and $\boldsymbol{T}f = Tf$ for $f \in \mathfrak{D}_T$. We shall retain the notation $\boldsymbol{T} \supseteqq T$ which we used for ordinary extensions (where $\mathbf{H} = \mathfrak{H}$).

The orthogonal projection of the "extension" space $\mathbf{H}$ onto its subspace $\mathfrak{H}$ will be denoted by $\boldsymbol{P}_{\mathfrak{H}}$ or simply by $\boldsymbol{P}$.

Among the extensions of a bounded linear transformation T in $\mathfrak{H}$ (with $\mathfrak{D}_T = \mathfrak{H}$) we shall consider in particular those which are of the form $\boldsymbol{PS}$ where $\boldsymbol{S}$ is a bounded linear transformation of an extension space $\mathbf{H}$. We express this relation

$$T \subseteq \boldsymbol{PS}$$

by saying that T is the *projection* of the transformation $\boldsymbol{S}$ onto $\mathfrak{H}$,[2] in symbols

(1) $$T = \mathrm{pr}_{\mathfrak{H}} \boldsymbol{S} \text{ or simply } T = \mathrm{pr}\, \boldsymbol{S}.$$

It is obvious that the relations $T_i = \mathrm{pr}\, \boldsymbol{S}_i\ (i = 1, 2)$ imply the relation

(2) $$a_1 T_1 + a_2 T_2 = \mathrm{pr}\,(a_1 \boldsymbol{S}_1 + a_2 \boldsymbol{S}_2)$$

(of course, $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are transformations in the *same* extension space $\mathbf{H}$). Relation (1) also implies that

(3) $$T^* = \mathrm{pr}\, \boldsymbol{S}^*.$$[3]

Finally, the uniform, strong, or weak convergence of a sequence $\{\boldsymbol{S}_n\}$ implies convergence of the same type for the sequence $\{T_n\}$ where $T_n = \mathrm{pr}\, \boldsymbol{S}_n$.[4]

If $\mathbf{H}$ and $\mathbf{H}'$ are two extension spaces of the same space $\mathfrak{H}$, $\boldsymbol{S}$ and $\boldsymbol{S}'$ are bounded linear transformations of $\mathbf{H}$ and $\mathbf{H}'$ respectively, then we shall say that the "structures" $\{\mathbf{H}, \boldsymbol{S}, \mathfrak{H}\}$ and $\{\mathbf{H}', \boldsymbol{S}', \mathfrak{H}\}$ are *isomorphic* if $\mathbf{H}$ can be mapped isometrically onto $\mathbf{H}'$ in such a way that the elements of the common subspace $\mathfrak{H}$ are left invariant and that $f \to f'$ implies $\boldsymbol{S}f \to \boldsymbol{S}'f'$.

If $\{\boldsymbol{S}_\omega\}_{\omega \in \Omega}$ and $\{\boldsymbol{S}'_\omega\}_{\omega \in \Omega}$ are two families of bounded linear transformations in $\mathbf{H}$ and $\mathbf{H}'$ respectively, we define the isomorphism of the "structures"

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[2] The terminology "T is the 'compression' of $\boldsymbol{S}$ in $\mathfrak{H}$, and $\boldsymbol{S}$ is the 'dilation' of T to $\boldsymbol{H}$" was proposed by HALMOS [**1**].

[3] In fact, we have

$$(f, T^*g) = (Tf, g) = (\boldsymbol{PSP}f, g) = (f, \boldsymbol{PS}^*\boldsymbol{P}g) = (f, \boldsymbol{PS}^*g)$$

for $f, g \in \mathfrak{H}$.

[4] In fact, we have $\|T_n - T_m\| \leqq \|\boldsymbol{S}_n - \boldsymbol{S}_m\|$, $\|(T_n - T_m)f\| \leqq \|(\boldsymbol{S}_n - \boldsymbol{S}_m)f\|$ for $f \in \mathfrak{H}$, and $((T_n - T_m)f, g) = ((\boldsymbol{S}_n - \boldsymbol{S}_m)f, g)$ for $f, g \in \mathfrak{H}$.

$\{\mathbf{H}, \boldsymbol{S}_\omega, \mathfrak{H}\}_{\omega\in\Omega}$ and $\{\mathbf{H}', \boldsymbol{S}'_\omega, \mathfrak{H}\}_{\omega\in\Omega}$ in the same manner by requiring that $f \to f'$ imply $\boldsymbol{S}_\omega f \to \boldsymbol{S}'_\omega f'$ for all $\omega \in \Omega$.

It is obvious that, from the point of view of extensions of transformations in $\mathfrak{H}$ which extend beyond $\mathfrak{H}$, two extensions which give rise to two isomorphic "structures" can be considered as identical.

In the sequel, when speaking of Hilbert spaces we shall mean both real and complex spaces. If we wish to distinguish between real and complex spaces, we shall say so explicitly. Of course, an extension space $\mathbf{H}$ of $\mathfrak{H}$ is always of the same type (real or complex) as $\mathfrak{H}$.

2. Generalized Spectral Families. NEUMARK'S Theorem

Extensions which extend beyond the given space were first investigated by M. A. NEUMARK [3, 4]; he investigated self-adjoint extensions of symmetric transformations in particular. If S is a symmetric transformation in the complex Hilbert space $\mathfrak{H}$ (with $\mathfrak{D}_S$ dense in $\mathfrak{H}$), we know that S cannot be extended to a self-adjoint transformation without extending beyond $\mathfrak{H}$ except when the deficiency indices $\mathfrak{m}$ and $\mathfrak{n}$ of S are equal. On the other hand, *there always exist self-adjoint extensions of S if one allows these extensions to extend beyond the space $\mathfrak{H}$.*

This is easily proved: Choose, in a Hilbert space $\mathfrak{H}'$, a symmetric transformation S' whose deficiency indices are $\mathfrak{n}$ and $\mathfrak{m}$, that is equal to the deficiency indices of S, but in the reverse order. One can take for example $\mathfrak{H}' = \mathfrak{H}$ and $S' = -S$. Having done this, we consider the product space $\mathbf{H} = \mathfrak{H} \times \mathfrak{H}'$ whose elements are pairs $\{f, f'\}$ $(f \in \mathfrak{H}, f' \in \mathfrak{H}')$ and in which the vector operations and metric are defined as follows:

$$c\{f, f'\} = \{cf, cf'\};\ \{f_1, f_1'\} + \{f_2, f_2'\} = \{f_1 + f_2, f_1' + f_2'\};$$
$$(\{f_1, f_1'\}, \{f_2, f_2'\}) = (f_1, f_2) + (f_1', f_2').$$

If we identify the element f in $\mathfrak{H}$ with the element $\{f, 0\}$ in $\mathbf{H}$, we embed $\mathfrak{H}$ in $\mathbf{H}$ as a subspace of the latter. The transformation

$$\boldsymbol{S}\{f, f'\} = \{Sf, S'f'\} \qquad (f \in \mathfrak{D}_S,\ f' \in \mathfrak{D}_{S'})$$

is then, as can easily be seen, a symmetric transformation in $\mathbf{H}$ having deficiency indices $\mathfrak{m} + \mathfrak{n}$, $\mathfrak{n} + \mathfrak{m}$. Consequently, $\boldsymbol{S}$ can be extended, without extending beyond $\mathbf{H}$, to a self-adjoint transformation $\boldsymbol{A}$ in $\mathbf{H}$. Since we have

$$S \subseteq \boldsymbol{S} \subseteq \boldsymbol{A}$$

(where the first extension is obtained by extension from $\mathfrak{H}$ to $\mathbf{H}$), we obtain a self-adjoint extension $\boldsymbol{A}$ of S.

Let

$$\boldsymbol{A} = \int_{-\infty}^{\infty} \lambda \, d\boldsymbol{E}_\lambda$$

be the spectral decomposition of $\boldsymbol{A}$. We have the relations

$$(Sf, g) = (\boldsymbol{A}f, \boldsymbol{P}g) = \int_{-\infty}^{\infty} \lambda \, d(\mathbf{E}_\lambda f, \boldsymbol{P}g) = \int_{-\infty}^{\infty} \lambda \, d(\boldsymbol{P}\mathbf{E}_\lambda f, g),$$

$$\|Sf\|^2 = \|\boldsymbol{A}f\|^2 = \int_{-\infty}^{\infty} \lambda^2 \, d(\mathbf{E}_\lambda f, f) = \int_{-\infty}^{\infty} \lambda^2 \, d(\boldsymbol{P}\mathbf{E}_\lambda f, f)$$

for $f \in \mathfrak{D}_S$, $g \in \mathfrak{H}$. Setting

$$B_\lambda = \operatorname{pr} \mathbf{E}_\lambda \tag{4}$$

we obtain a family $\{B_\lambda\}_{-\infty<\lambda<\infty}$ of bounded self-adjoint transformations in the space $\mathfrak{H}$, which have the following properties:

a) $B_\lambda \leqq B_\mu$ for $\lambda < \mu$;

b) $B_{\lambda+0} = B_\lambda$;

c) $B_\lambda \to O$ as $\lambda \to -\infty$; $B_\lambda \to I$ as $\lambda \to +\infty$.

Every one-parameter family of bounded self-adjoint transformations which have these properties will be called a *generalized spectral family*. If this family consists of projections (which are then, as a consquence of a), mutually permutable), then we have an ordinary spectral family.

According to what we just proved, we can assign to each symmetric transformation S in $\mathfrak{H}$ a generalized spectral family $\{B_\lambda\}$ in such a way that the equations

$$(Sf, g) = \int_{\infty}^{\infty} \lambda \, d(B_\lambda f, g), \quad \|Sf\|^2 = \int_{\infty}^{\infty} \lambda^2 \, d(B_\lambda f, f) \tag{5}$$

are satisfied for $f \in \mathfrak{D}_S$, $g \in \mathfrak{H}$ (where the integral in the second equation can also converge for certain f which do not belong to $\mathfrak{D}_S$).

The question arises: Can every generalized spectral family which belongs to S, that is, which satisfies equations (5), be obtained as the projection of the spectral family of a self-adjoint extension $\boldsymbol{A}$ of S? The answer is in the affirmative. Namely, we have the following theorem.

THEOREM I (NEUMARK [4. 5]). *Every generalized spectral family* $\{B_\lambda\}$ *can be represented in the form* (4), *as the projection of an ordinary spectral family* $\{\mathbf{E}_\lambda\}$. *One can even require the extension space* $\mathbf{H}$ *to be minimal in the sense that it be spanned by the elements of the form* $\mathbf{E}_\lambda f$ *where* $f \in \mathfrak{H}$, $-\infty < \lambda < \infty$; *in this case, the structure* $\{\mathbf{H}, \mathbf{E}_\lambda, \mathfrak{H}\}_{-\infty<\lambda<\infty}$ *is determined to within an isomorphism.*

We shall prove this theorem in Sec. 7 as a corollary to the "principal theorem" (Sec. 6).

We observe that if $\mathbf{H}$ is minimal, every interval of constancy of B_λ is also an interval of constancy for $\mathbf{E}_\lambda$. In fact, if $a \leqq \lambda < b$ is an interval of constancy of B_λ, we have

$$\|\boldsymbol{E}_\lambda - \boldsymbol{E}_a)\boldsymbol{E}_\mu f\|^2 = \|(\boldsymbol{E}_{\min\{\lambda,\mu\}} - \boldsymbol{E}_{\min\{a,\mu\}})f\|^2 = ((\boldsymbol{E}_{\min\{\lambda,\mu\}} - \boldsymbol{E}_{\min\{a,\mu\}})f, f) =$$

$$= (\boldsymbol{P}(\boldsymbol{E}_{\min\{\lambda,\mu\}} - \boldsymbol{E}_{\min\{a,\mu\}})f, f) = ((B_{\min\{\lambda,\mu\}} - B_{\min\{a,\mu\}})f, f) = 0$$

for $f \in \mathfrak{H}$, $a \leqq \lambda < b$, and μ an arbitrary real number; hence

$$(\boldsymbol{E}_\lambda - \boldsymbol{E}_a) g = 0$$

for every element g of the form $\boldsymbol{E}_\mu f\ (f \in \mathfrak{H})$. Since these elements g span the space $\mathbf{H}$, we have $\boldsymbol{E}_\lambda - \boldsymbol{E}_a = \boldsymbol{O}$, $\boldsymbol{E}_\lambda = \boldsymbol{E}_a$, which completes the proof of the theorem.

The simplest case of the Neumark theorem occurs when the family $\{B_\lambda\}$ is generated by a self-adjoint transformation A such that $O \leqq A \leqq I$, in the following manner:

$$B_\lambda = O \text{ for } \lambda < a,\ B_\lambda = A \text{ for } a \leqq \lambda < b,\ B_\lambda = I \text{ for } \lambda \geqq b.$$

We thus obtain the following corollary.

COROLLARY. *Every self-adjoint transformation A in the Hilbert space $\mathfrak{H}$, such that $O \leqq A \leqq I$, can be represented in the form*

$$A = \operatorname{pr} \boldsymbol{Q}$$

where $\boldsymbol{Q}$ is a projection in an extension space $\mathbf{H}$. *In brief: A is the projection of a projection.*

This corollary can also be proved directly without recourse to the general Neumark theorem. The following construction is due to E. A. MICHAEL.[5]

Consider the product space $\mathbf{H} = \mathfrak{H} \times \mathfrak{H}$; by identifying the element f in $\mathfrak{H}$ with the element $\{f, 0\}$ in $\mathbf{H}$, we embed $\mathfrak{H}$ in $\mathbf{H}$ as a subspace of the latter. If we write the elements $\mathbf{H}$ as one-column matrices $\begin{pmatrix} f \\ g \end{pmatrix}$, then every bounded linear transformation $\boldsymbol{T}$ in $\mathbf{H}$ can be represented in the form of a matrix

$$\boldsymbol{T} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \tag{6}$$

whose elements T_{ik} are bounded linear transformations in $\mathfrak{H}$. It is easily verified that the matrix addition and multiplication of the corresponding matrices correspond to the addition and multiplication of transformations. Moreover, relation (6) implies that

$$\boldsymbol{T}^* = \begin{pmatrix} T_{11}^* & T_{21}^* \\ T_{12}^* & T_{22}^* \end{pmatrix}.$$

Finally, we have

$$T = \operatorname{pr} \boldsymbol{T}$$

if and only if

$$T_{11} = T.$$

This done, we consider the transformation

$$\boldsymbol{Q} = \begin{pmatrix} A & B \\ B & I - A \end{pmatrix} \quad \text{with} \quad B = [A(I-A)]^{\frac{1}{2}}.$$

---

[5] See HALMOS [1].

It is clear that Q is self-adjoint and that $A = \text{pr}\, Q$. It remains only to show that $Q^2 = Q$, which is easily done by calculating the square of the matrix Q.

The following theorem is another, less special, consequence of the Neumark theorem.

THEOREM. *Every finite or infinite sequence $\{A_n\}$ of bounded self-adjoint transformations in the Hilbert space $\mathfrak{H}$ such that*

$$A_n \geqq O, \quad \Sigma A_n = I$$

can be represented in the form

$$A_n = \text{pr}\, \boldsymbol{Q}_n \qquad (n = 1, 2, \ldots),$$

where $\{\boldsymbol{Q}_n\}$ is a sequence of projections of an extension space **H** *for which*

$$\boldsymbol{Q}_n \boldsymbol{Q}_m = \boldsymbol{O} \quad (m \neq n), \quad \Sigma \boldsymbol{Q}_n = \boldsymbol{I}.$$

In fact, one has only to apply the Neumark theorem to the generalized spectral family $\{B_\lambda\}$ defined by

$$B_\lambda = \sum_{n \leqq \lambda} A_n.$$

If $\{\boldsymbol{E}_\lambda\}$ is an ordinary spectral family in a minimal extension space such that $B_\lambda = \text{pr}\, \boldsymbol{E}_\lambda$, the function $\boldsymbol{E}_\lambda$ of λ increases only at the points n where it has the jumps

$$\boldsymbol{Q}_n = \boldsymbol{E}_n - \boldsymbol{E}_{n-0};$$

these transformations $\boldsymbol{Q}_n$ satisfy the requirements of the theorem.

This theorem in its turn has the following theorem as a consequence.

THEOREM. *Every finite or infinite sequence $\{T_n\}$ of bounded linear transformations in the complex Hilbert space $\mathfrak{H}$ can be represented by means of a sequence $\{\boldsymbol{N}_n\}$ of bounded normal transformations in an extension space* **H** *in the form*

$$T_n = \text{pr}\, \boldsymbol{N}_n \qquad (n = 1, 2, \ldots),$$

where the $\boldsymbol{N}_n$ are pairwise doubly permutable.[6] *If any of the transformations T_n are self-adjoint, the corresponding $\boldsymbol{N}_n$ can also be chosen to be self-adjoint.*

We first consider the case where all the transformations T_n are self-adjoint. If m_n and M_n are the greatest lower and least upper bounds of T_n, we set

$$A_n = \frac{1}{2^n(M_n - m_n + 1)}(T_n - m_n I) \qquad (n = 1, 2, \ldots);$$

then we obviously have

$$A_n \geqq O, \quad \sum_n A_n \leqq I.$$

---

[6] We shall say that two bounded linear transformations T_1, T_2 are *doubly permutable* if T_1 is permutable with T_2 and with T_2^* (and then T_2 is permutable with T_1 and with T_1^*). Furthermore, for two *normal* transformations, simple permutability implies double permutability; it even suffices to assume that one of the two transformations is normal (FUGLEDE [1]; also see HALMOS [2]).

If we again set

$$A = I - \sum_n A_n$$

we obtain a sequence $A, A_1, A_2, \ldots$ of transformations which satisfies the hypotheses of the preceding theorem, and which, consequently, can be represented in the form

$$A_n = \text{pr}\, \boldsymbol{Q}_n \qquad (n = 1, 2, \ldots)$$

in terms of the projections $\boldsymbol{Q}_n$, which are pairwise orthogonal (and consequently permutable). It follows that

$$T_n = \text{pr}\, \boldsymbol{S}_n \qquad (n = 1, 2, \ldots)$$

with

$$\boldsymbol{S}_n = m_n \boldsymbol{I} + 2^n (M_n - m_n + 1) \boldsymbol{Q}_n,$$

where the transformations $\boldsymbol{S}_n$ are self-adjoint and mutually permutable.

The general case is reducible to the particular case of self-adjoint transformations by replacing each transformation T_n in the given sequence by the two self-adjoint transformations $\text{Re}\, T_n$ and $\text{Im}\, T_n$. In fact, since the representation

$$\text{Re}\, T_n = \text{pr}\, \boldsymbol{S}_{2n}, \quad \text{Im}\, T_n = \text{pr}\, \boldsymbol{S}_{2n+1} \qquad (n = 1, 2, \ldots)$$

is possible by means of bounded self-adjoint pairwise permutable transformations $\boldsymbol{S}_i$, the representation

$$T_n = \text{pr}\, \boldsymbol{N}_n \qquad (n = 1, 2, \ldots)$$

follows from this by means of the normal pairwise doubly permutable transformations $\boldsymbol{N}_n = \boldsymbol{S}_{2n} + i\boldsymbol{S}_{2n+1}$. For a self-adjoint T_n, we have $T_n = \text{Re}\, T_n$, and we can then choose $\boldsymbol{S}_{2n+1} = \boldsymbol{O}$ and hence $\boldsymbol{N}_n = \boldsymbol{S}_{2n}$.

3. Sequences of Moments

1. The following theorem is closely related to Theorem I.

THEOREM II (SZ.-NAGY [9]). *Suppose $\{A_n\}$ $(n = 0, 1, \ldots)$ is a sequence of bounded self-adjoint transformations in the Hilbert space $\mathfrak{H}$ satisfying the following conditions:*

(α_M) *for every polynomial*

$$a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n$$

with real coefficients which assumes non-negative values in the interval $-M \leqq \lambda \leqq M$, we have

$$a_0 A_0 + a_1 A_1 + a_2 A_2 + \cdots + a_n A_n \geqq O\,;$$

(β) $A_0 = I$.

Then there exists a self-adjoint transformation $\boldsymbol{A}$ in an extension space **H** *such that*

(7) $$A_n = \text{pr}\, \boldsymbol{A}^n \qquad (n = 0, 1, \ldots).$$

Furthermore, one can require that **H** *be minimal in the sense that it be spanned by elements of the form $A^n f$ where $f \in \mathfrak{H}$ and $n = 0, 1, \ldots$; in this case, the structure $\{\mathbf{H}, A, \mathfrak{H}\}$ is determined to within an isomorphism, and we have*

$$\|A\| \leqq M.$$

We observe that if $\{B_\lambda\}$ is a generalized spectral family on the interval $[-M, M]$ (that is, $B_\lambda = O$ for $\lambda < -M$ and $B_\lambda = I$ for $\lambda \geqq M$), the transformations

$$A_n = \int_{-M-0}^{M} \lambda^n \, dB_\lambda \qquad (n = 0, 1, \ldots) \tag{8}$$

satisfy conditions (α_M) and (β). Conversely, if these conditions are satisfied, the sequence $\{A_n\}$ has an integral decomposition of the form (8) with $\{B_\lambda\}$ on $[-M, M]$. This clearly follows from Theorem II if we make use of the spectral decomposition of the transformation A. But one can also prove (8) directly, without recourse to Theorem II.

In fact, the correspondence between the polynomials

$$p(\lambda) = a_0 + a_1 \lambda + a_2 \lambda^2 + \cdots + a_n \lambda^n$$

and the self-adjoint transformations

$$A(p) = a_0 I + a_1 A_1 + a_2 A_2 + \cdots + a_n A_n$$

which is *homogeneous*, *additive*, and *of positive type* with respect to the interval $-M \leqq \lambda \leqq M$, can be extended, with preservation of these properties, to a vaster class of functions which comprises, among others, the discontinuous functions

$$e_\mu(\lambda) = \begin{cases} 1 \text{ for } \lambda \leqq \mu, \\ 0 \text{ for } \lambda > \mu, \end{cases}$$

and then we obtain representation (8) by setting

$$B_\mu = A(e_\mu).$$

We have only to repeat verbatim the line of argument of one of the usual proofs of the spectral decomposition of a bounded self-adjoint transformation A,[7] letting A_n play the role of A^n. The only difference is that now the correspondence $p(\lambda) \to p(A)$ and its extension are no longer multiplicative, and that consequently the relation $e_\mu^2(\lambda) \equiv e_\mu(\lambda)$ does not imply that B_μ^2 is equal to B_μ, and hence that B_μ is in general not a projection.

According to Theorem I, $\{B_\lambda\}$ is the projection of an ordinary spectral family $\{\mathbf{E}_\lambda\}$, which one can choose in such a way that it is also on $[-M, M]$, and then (7) follows from (8) by setting

$$A = \int_{-M-0}^{M} \lambda \, d\mathbf{E}_\lambda .$$

---

[7] See Secs. 106, 107.

We shall return to this theorem later (Sec. 8) and prove it as a corollary to the "principal theorem" (Sec. 6).

2. If we replace condition (β) by the less restrictive condition

$$(\beta') \qquad A_0 \leqq I,$$

then representation (7) of the sequence $\{A_n\}$ will still be possible, if only starting from $n=1$.[8] Everything reduces to showing that if the sequence

$$\{A_0, A_1, A_2, \ldots\}$$

satisfies conditions (α_M) and (β'), the sequence

$$\{I, A_1, A_2, \ldots\}$$

satisfies condition (α_M). But, if $p(\lambda)=a_0+a_1\lambda+\ldots+a_n\lambda^n \geqq 0$ in $[-M, M]$, we have in particular that $p(0)=a_0 \geqq 0$; since by assumption, $a_0A_0+a_1A_1+\ldots+a_nA_n \geqq O$ and $I-A_0 \geqq O$, it follows that

$$a_0 I + a_1 A_1 + \cdots + a_n A_n = a_0(I - A_0) + a_0 A_0 + a_1 A_1 + \cdots + a_n A_n \geqq O.$$

One of the most interesting consequences of the representation

$$A_n = \operatorname{pr} \boldsymbol{A}^n \qquad (n=1, 2, \ldots)$$

is the following. We have

$$\begin{aligned}(A_2 f, f) &= (\boldsymbol{P}\boldsymbol{A}^2 f, f) = (\boldsymbol{A}^2 f, f) = \|\boldsymbol{A} f\|^2 \geqq \\ &\geqq \|\boldsymbol{P}\boldsymbol{A} f\|^2 = (\boldsymbol{A}\boldsymbol{P}\boldsymbol{A} f, f) = (\boldsymbol{P}\boldsymbol{A}\boldsymbol{P}\boldsymbol{A} f, f) = (A_1^2 f, f)\end{aligned}$$

for all $f \in \mathfrak{H}$, where equality holds if, and only if,

$$\boldsymbol{A} f = \boldsymbol{P}\boldsymbol{A} f = A_1 f.$$

If this case occurs for all $f \in \mathfrak{H}$, we have

$$\begin{aligned}\boldsymbol{A}^2 f &= \boldsymbol{A}(\boldsymbol{A} f) = \boldsymbol{A}(A_1 f) = A_1(A_1 f) = A_1^2 f,\\ \boldsymbol{A}^3 f &= \boldsymbol{A}(\boldsymbol{A}^2 f) = \boldsymbol{A}(A_1^2 f) = A_1(A_1^2 f) = A_1^3 f, \text{ etc.}\end{aligned}$$

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[8] Formula (7) in Theorem II can be replaced by another which is valid under the single condition (α_M) and for $n=0, 1, \ldots$. Namely, we have that

$$A_n f = A_0^{\frac{1}{2}} \boldsymbol{P}\boldsymbol{A}^n A_0^{\frac{1}{2}} f \qquad (f \in \mathfrak{H};\ n=0, 1, \ldots),$$

where $\boldsymbol{A}$ is a self-adjoint transformation of a suitable extension space, $\|\boldsymbol{A}\| \leqq M$. (The equality $A_0 \geqq O$ and hence the existence of the square root $A_0^{\frac{1}{2}} \geqq O$ are consequences of the condition (α_M) if we apply it to $p(\lambda) \equiv 1$.) This is obtained immediately when A_0 admits even a positive greatest lower bound, because then the sequence of transformations

$$\tilde{A}_n = A_0^{-\frac{1}{2}} A_n A_0^{-\frac{1}{2}} \qquad (n=0, 1, \ldots)$$

satisfies conditions (α_M) and (β). The case when A_0 does not have a positive greatest lower bound requires a slightly more refined investigation (see Sz.-Nagy **[9]**).

and hence

$$A_n f = \boldsymbol{P} A'' f = A_1^n f \qquad (n = 1, 2, \ldots).$$

We have thus obtained the following result.

If the sequence $A_0, A_1, A_2, \ldots$ of bounded self-adjoint transformations in the Hilbert space $\mathfrak{H}$ satisfies hypotheses (α_M) and (β'), then the inequality

$$A_1^2 \leqq A_2 \tag{9}$$

holds, where equality occurs if, and only if, $A_n = A_1^n$ $(n = 1, 2, \ldots)$.

Inequality (9) is due to R. V. Kadison [1] who proved it differently and used it in his researches on algebraic invariants of operator algebras.

Moreover, one can also omit hypothesis (β'), and then the following inequality

$$A_1^2 \leqq \|A_0\| A_2 \tag{10}$$

is obtained; in fact, we have only to apply inequality (9) to the sequence $\{\|A_0\|^{-1} A_n\}$.

4. Contractions in Hilbert Space

1. Whereas the projections of bounded self-adjoint transformations are also self-adjoint, the projections of unitary transformations[9] are already of a more general type. In order that $T = \operatorname{pr} \boldsymbol{U}$, with $\boldsymbol{U}$ unitary, it is necessary that

$$\|Tf\| = \|\boldsymbol{P}\boldsymbol{U}f\| \leqq \|\boldsymbol{U}f\| = \|f\|$$

for all $f \in \mathfrak{H}$, that is, $\|T\| \leqq 1$, and hence the transformation T must be a *contraction*.

But, this condition is not only necessary but also sufficient.

THEOREM. *Every contraction T in the Hilbert space $\mathfrak{H}$ can be represented in an extension space $\mathbf{H}$ as the projection of a unitary transformation $\boldsymbol{U}$ onto $\mathfrak{H}$.*

The theorem, and the following simple construction of $\boldsymbol{U}$, are due to Halmos [1]. As in Sec. 2, let us consider the product space $\mathbf{H} = \mathfrak{H} \times \mathfrak{H}$ and the following transformation of $\mathbf{H}$:

$$\boldsymbol{U} = \begin{pmatrix} T & S \\ -Z & T^* \end{pmatrix} \text{ where } S = (I - TT^*)^{\frac{1}{2}},\ Z = (I - T^*T)^{\frac{1}{2}}.\text{[10]} \tag{11}$$

The relation $T = \operatorname{pr} \boldsymbol{U}$ is obvious. We shall show that $\boldsymbol{U}$ is unitary, or, what

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[9] It is usual to speak of *unitary* transformations only in the case of a complex Hilbert space; their analogues in the case of real Hilbert space are called *orthogonal*. As a matter of convenience, we agree to say "unitary" in both cases. Hence, the linear transformation T in the Hilbert space $\mathfrak{H}$ is *unitary* if it maps the space $\mathfrak{H}$ isometrically onto itself, or, what amounts to the same thing, if $T^*T = TT^* = I$.

[10] Since $\|T\| \leqq 1$, we have $O \leqq I - TT^* \leqq I$ and $O \leqq I - T^*T \leqq I$.

amounts to the same thing, that $\boldsymbol{U}^*\boldsymbol{U}$ and $\boldsymbol{U}\boldsymbol{U}^*$ are equal to the identity transformation $\boldsymbol{I}$ in $\mathbf{H}$. Since S and Z are self-adjoint, we have

$$\boldsymbol{U}^*\boldsymbol{U}=\begin{pmatrix} T^* & -Z \\ S & T\end{pmatrix}\begin{pmatrix} T & S \\ -Z & T^*\end{pmatrix}=\begin{pmatrix} T^*T+Z^2 & T^*S-ZT \\ ST-TZ & S^2+TT^*\end{pmatrix},$$

$$\boldsymbol{U}\boldsymbol{U}^*=\begin{pmatrix} T & S \\ -Z & T^*\end{pmatrix}\begin{pmatrix} T^* & -Z \\ S & T\end{pmatrix}=\begin{pmatrix} TT^*+S^2 & -TZ+ST \\ -ZT^*+TS & Z^2+T^*T\end{pmatrix}.$$

Since $Z^2=I-T^*T$, $S^2=I-TT^*$, the diagonal elements of the product matrices are all equal to I. It remains to show that the other elements are equal to O, i.e. that

$$ST=TZ \tag{12}$$

(the equation $T^*S=ZT^*$ follows from this by passing over to the adjoints of both members of (12)).

But, we have

$$S^2T=(I-TT^*)T=T-TT^*T=T(I-T^*T)=TZ^2,$$

from which it follows by complete induction that

$$S^{2n}T=TZ^{2n} \quad \text{for} \quad n=0,1,2,\ldots.$$

Then we also have

$$p(S^2)T=Tp(T^2)$$

for every polynomial $p(\lambda)$. Since S and Z are the positive square roots of S^2 and Z^2 respectively, there exists a sequence of polynomials $p_n(\lambda)$ such that

$$p_n(S^2)\to S,\quad p_n(Z^2)\to Z.$$ [11]

Now (12) follows from the equation

$$p_n(S^2)T=Tp_n(Z^2)$$

by passing to the limit as $n\to\infty$.

This completes the proof of the theorem.

2. The relation between transformations $\boldsymbol{S}$ in an extension space $\mathbf{H}$ of the space $\mathfrak{H}$ and their projections $T=\operatorname{pr}\boldsymbol{S}$ onto $\mathfrak{H}$ is not multiplicative in general, that is, the equations $T_1=\operatorname{pr}\boldsymbol{S}_1$, $T_2=\operatorname{pr}\boldsymbol{S}_2$ do not in general imply $T_1T_2=\operatorname{pr}\boldsymbol{S}_1\boldsymbol{S}_2$. For example, if we consider the transformation $\boldsymbol{U}$ constructed according to formula (11), we have $\operatorname{pr}\boldsymbol{U}^2=T^2-SZ$, which in general is not equal to T^2.

The question arises: Is it possible to find, in a suitable extension space, a unitary transformation $\boldsymbol{U}$ such that the powers of the contraction T (which are themselves contractions) are at the same time equal to the projections onto $\mathfrak{H}$ of the corresponding powers of $\boldsymbol{U}$?

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[11] See Sec. 104.

If we are dealing with only a finite number of powers,

$$T, T^2, \ldots, T^k,$$

then the problem can be solved in the affirmative in a rather simple manner, by suitably generalizing the immediately preceding construction.

Let us consider the product space $\mathbf{H} = \mathfrak{H} \times \cdots \times \mathfrak{H}$, with $k+1$ factors, whose elements are ordered $(k+1)$-tuples $\{f_1, \ldots, f_{k+1}\}$ of elements in $\mathfrak{H}$ and in which the vector operations and metric are defined in the usual way:

$$c\{f_1, \ldots, f_{k+1}\} = \{cf_1, \ldots, cf_{k+1}\},$$
$$\{f_1, \ldots, f_{k+1}\} + \{g_1, \ldots, g_{k+1}\} = \{f_1 + g_1, \ldots, f_{k+1} + g_{k+1}\},$$
$$(\{f_1, \ldots, f_{k+1}\}, \{g_1, \ldots, g_{k+1}\}) = (f_1, g_1) + \cdots + (f_{k+1}, g_{k+1}).$$

We embed $\mathfrak{H}$ in $\mathbf{H}$ as a subspace of the latter by identifying the element f in $\mathfrak{H}$ with the element $\{f, 0, \ldots, 0\}$ in $\mathbf{H}$. The bounded linear transformations $\boldsymbol{T}$ in $\mathbf{H}$ will be represented by matrices (T_{ij}) with $k+1$ rows and $k+1$ columns, all of whose elements T_{ij} are bounded linear transformations in $\mathfrak{H}$. We have $T = \operatorname{pr} \boldsymbol{T}$ if and only if $T_{11} = T$.

Let us now consider the following transformation in $\mathbf{H}$:[12]

$$\boldsymbol{U} = \left.\begin{pmatrix} T & S & O & O & \ldots & O \\ O & O & -I & O & \ldots & O \\ O & O & O & -I & \ldots & O \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ O & O & O & \ldots & O & -I \\ -Z & T^* & O & \ldots & O & O \end{pmatrix}\right\} k+1 \text{ rows and columns,}$$

where S and Z have the same meaning as in the foregoing construction.[13]

The transformation $\boldsymbol{U}$ is unitary. This is proved in the same way as above, by a direct calculation of the matrices $\boldsymbol{U}^*\boldsymbol{U}$, $\boldsymbol{U}\boldsymbol{U}^*$. In order to prove the relations

$$T^n = \operatorname{pr} \boldsymbol{U}^n \qquad (n = 1, 2, \ldots, k),$$

[12] The following construction is a modification of that given by EGERVÁRY [1] for the case of a finite-dimensional space.

[13] The $-I$'s could be replaced by $+I$'s, but the choice of the minus sign has the following advantage: Suppose $\mathfrak{H}$ is real and finite-dimensional and represent the transformations T, T^*, S, Z, O and I by their matrices with respect to an orthogonal basis in $\mathfrak{H}$. $\boldsymbol{U}$ will then be an ordinary hypermatrix whose elements are real numbers. Because of the minus signs, the determinant of $\boldsymbol{U}$ is equal to the determinant d of the matrix $\begin{pmatrix} T & S \\ -Z & T^* \end{pmatrix}$. Since the latter matrix is orthogonal, we have $d = \pm 1$. But d depends continuously on T and since the contractions form a convex (and hence connected) set, and since furthermore $d = +1$ for $T = I$, we necessarily have $d = +1$ for all the contractions T. Hence $\boldsymbol{U}$ is an orthogonal transformation which preserves the orientation of the space, that is, it is a *rotation*.

we must calculate the element in the matrix $\boldsymbol{U}^n$ having indices 1, 1 and then note that the latter is equal to T^n for $n = 1, \ldots, k$. We shall even prove more, namely that the first row in the matrix $\boldsymbol{U}^n$ $(n = 1, \ldots, k)$ is the following:

$$(T^n,\ T^{n-1}S,\ -T^{n-2}S^2,\ T^{n-3}S^3,\ -T^{n-4}S^4, \ldots, (-1)^{n-1}S^n, \overbrace{O, \ldots, O}^{k-n}).$$

This proposition is obvious for $n = 1$, and we prove it true for $n+1$, assuming it true for n $(n \leqq k-1)$ by calculating the matrix $\boldsymbol{U}^{n+1}$ as the matrix product $\boldsymbol{U}^n \cdot \boldsymbol{U}$. We have thus proved the following theorem.

THEOREM. *If T is a contraction in the Hilbert space $\mathfrak{H}$, then there exists a unitary transformation* $\mathbf{U}$ *in an extension space* $\mathbf{H}$ *such that*

$$T^n = \operatorname{pr} \boldsymbol{U}^n,$$

$n = 0, 1, \ldots, k$ *(the case $n = 0$ is trivial), for every given natural number k. The product space $\mathfrak{H} \times \cdots \times \mathfrak{H}$ with $k+1$ factors*[14] *can be taken for* $\mathbf{H}$.

3. It is important in the above construction that k is a finite number. However, the theorem is also true for $k = \infty$.

THEOREM III (SZ.-NAGY [**10, 11**]). *If T is a contraction in the Hilbert space $\mathfrak{H}$, then there exists a unitary transformation* $\boldsymbol{U}$ *of an extension space* $\mathbf{H}$ *such that the relation*

$$T^n = \operatorname{pr} \boldsymbol{U}^n$$

is valid for $n = 0, 1, 2, \ldots$. Furthermore, one can require that the space $\mathbf{H}$ *be minimal in the sense that it is spanned by the elements of the form $\boldsymbol{U}^n f$ where $f \in \mathfrak{H}$ and $n = 0, \pm 1, \pm 2, \ldots$; in this case, the structure $\{\mathbf{H}, \boldsymbol{U}, \mathfrak{H}\}$ is determined to within an isomorphism.*

An analogous theorem is true for *semi-groups* and *one-parameter semi-groups* of contractions, that is, for families $\{T_t\}$ of contractions (where $0 \leqq t < \infty$ or $-\infty \leqq t < \infty$, according to the case at hand) such that

$$T_0 = I, \quad T_{t_1} T_{t_2} = T_{t_1 + t_2},$$

and for which one assumes further that T_t depends strongly or weakly continuously on t; weak continuity means that $(T_t f, g)$ is a continuous numerical-valued function of t for every pair f, g of elements in $\mathfrak{H}$. The theorem in question is the following.

THEOREM IV (SZ.-NAGY [**10, 11**]). *If $\{T_t\}_{t \geq 0}$ is a weakly continuous one-parameter semi-group of contractions in the Hilbert space $\mathfrak{H}$, then there exists a one-parameter group $\{\boldsymbol{U}_t\}_{-\infty < t < \infty}$ of unitary transformations in an extension space* $\mathbf{H}$, *such that*

$$T_t = \operatorname{pr} \boldsymbol{U}_t \quad \text{for} \quad t \geqq 0.$$

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[14] The last proposition is important only in the case where the space $\mathfrak{H}$ is finite-dimensional.

Furthermore, one can require that the space $\mathbf{H}$ *be minimal in the sense that it is spanned by elements of the form* $\boldsymbol{U}_t f$, *where* $f \in \mathfrak{H}$ *and* $-\infty < t < \infty$; *in this case,* $\boldsymbol{U}_t$ *is strongly continuous and the structure* $\{\mathbf{H}, \boldsymbol{U}_t, \mathfrak{H}\}_{-\infty<t<\infty}$ *is determined to within an isomorphism.*

These two theorems can be generalized to discrete or continuous semi-groups with several generators. We shall formulate only the following generalization of Theorem III.

THEOREM V. *Suppose* $\{T^{(\rho)}\}_{\rho\in R}$ *is a system of pairwise doubly permutable contractions in the Hilbert space* $\mathfrak{H}$. *There exists, in an extension space* $\mathbf{H}$, *a system* $\{\boldsymbol{U}^{(\rho)}\}_{\rho\in R}$ *of pairwise permutable unitary transformations such that*

$$\prod_{i=1}^{r}[T^{(\rho_i)}]^{n_i} = \mathrm{pr} \prod_{i=1}^{r}[\boldsymbol{U}^{(\rho_i)}]^{n_i}$$

for arbitrary $\rho_i \in R$ *and integers* n_i, *provided the factor* $[T^{(\rho_i)}]^{n_i}$ *is replaced by* $[T^{(\rho_i)*}]^{-n_i}$ *when* $n_i < 0$. *Moreover, one can require that the space* $\mathbf{H}$ *be minimal in the sense that it be spanned by the elements of the form* $\prod_{i=1}^{r}[\boldsymbol{U}^{(\rho_i)}]^{n_i} f$ *where* $f \in \mathfrak{H}$; *in this case, the structure* $\{\mathbf{H}, \boldsymbol{U}^{(\rho)}, \mathfrak{H}\}_{\rho\in R}$ *is determined to within an isomorphism.*

We shall prove Theorems III and V in Sec. 9.

3. We now give several applications of these theorems; T will denote a contraction in a complex Hilbert space $\mathfrak{H}$ and $\{T_t\}$ is a weakly continuous one-parameter semi-group of contractions in $\mathfrak{H}$.

a) Invariant elements. *If the element* f *is invariant with respect to* T, *then it is also invariant with respect to* T^*.[15]

Proof. We have $T = \mathrm{pr}\,\boldsymbol{U}$, with $\boldsymbol{U}$ unitary, from which it follows that $T^* = \mathrm{pr}\,\boldsymbol{U}^* = \mathrm{pr}\,\boldsymbol{U}^{-1}$. The equations $f = Tf = \boldsymbol{P}\boldsymbol{U}f$, $\|\boldsymbol{U}f\| = \|f\|$ imply that $\boldsymbol{U}f = f$. Hence we have $f = \boldsymbol{U}^{-1}f = \boldsymbol{P}\boldsymbol{U}^{-1}f = T^*f$, which completes the proof of the theorem.

b) Ergodic theorems. *For all* $f \in \mathfrak{H}$ *the limits*

$$\lim_{\substack{n>m\geq 0\\ n-m\to\infty}} \frac{1}{n-m}\sum_{k=m}^{n-1} T^k f$$

and

$$\lim_{\substack{\nu>\mu\geq 0\\ \nu-\mu\to\infty}} \frac{1}{\nu-\mu}\int_{\mu}^{\nu} T_t f\,dt$$

exist in the sense of strong convergence of elements, where the integral is defined as the strong limit of sums of Riemann type.[16]

Proof. By Theorems III and IV, we have $T^k = \mathrm{pr}\,\boldsymbol{U}^k$ $(k = 0, 1, \ldots)$ and $T_t = \mathrm{pr}\,\boldsymbol{U}_t$ $(t \geq 0)$ with $\boldsymbol{U}_t$ strongly continuous; hence T_t is also strongly continuous. For $f \in \mathfrak{H}$, we have

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[15] See Sec. 144.

[16] See Sec. 144.

$$\sum_{m}^{n-1} T^k f = \boldsymbol{P} \sum_{m}^{n-1} \boldsymbol{U}^k f$$

and

$$\int_{\mu}^{\nu} T_t f\,dt = \boldsymbol{P} \int_{\mu}^{\nu} \boldsymbol{U}_t f\,dt,$$

respectively, and the propositions thus follow from the ergodic theorems of J. VON NEUMANN on unitary transformations.

DUNFORD'S ergodic theorem on several permutable[17] contractions can be reduced in an analogous manner, by Theorem V, to the particular case of unitary transformations, but this only under the additional condition that these contractions be *doubly* permutable.

c) Theorems of VON NEUMANN and HEINZ.[18] *Suppose*

$$u(z) = c_0 + c_1 z + \cdots + c_n z^n + \cdots$$

is a power series in the complex variable z with

$$|c_0| + |c_1| + \cdots + |c_n| + \cdots < \infty. \tag{13}$$

Set

$$u(T) = c_0 I + c_1 T + \ldots + c_n T^n + \ldots .^{19}$$

If the function $u(z)$ satisfies one or the other of the inequalities

$$|u(z)| \leqq 1, \qquad \operatorname{Re} u(z) \geqq 0, \text{ with } |z| \leqq 1,$$

then we have

$$\|u(T)\| \leqq 1, \qquad \operatorname{Re} u(T) \geqq O,$$

respectively.

Proof: It follows from the representation of powers: $T^k = \operatorname{pr} \boldsymbol{U}^k$ $(k = 0, 1, \ldots)$ that

$$u(T) = \operatorname{pr} u(\boldsymbol{U}).$$

Let

$$\boldsymbol{U} = \int_0^{2\pi} e^{i\lambda}\, d\boldsymbol{E}_\lambda$$

be the spectral decomposition of the unitary transformation $\boldsymbol{U}$; we then have

$$\|u(T)f\|^2 = \|\boldsymbol{P} u(\boldsymbol{U})f\|^2 \leqq \|u(\boldsymbol{U})f\|^2 = \int_0^{2\pi} |u(e^{i\lambda})|^2 d(\boldsymbol{E}_\lambda f, f),$$

$$\operatorname{Re}(u(T)f, f) = \operatorname{Re}(\boldsymbol{P}\, u(\boldsymbol{U})f, f) = \operatorname{Re}(u(\boldsymbol{U})f, f) = \int_0^{2\pi} \operatorname{Re} u(e^{i\lambda})\, d(\boldsymbol{E}_\lambda f, f)$$

[17] See Sec. 145.

[18] See Sec. 153; here they are stated in a slightly generalized form. These theorems are valid in a complex Hilbert space.

[19] This series converges in norm because of (13).

for $f \in \mathfrak{H}$. The above propositions follow in an obvious manner from these formulas.

d) *Let*

$$p(\theta) = \sum_k a_k e^{it_k \theta}$$

be a trigonometric series with arbitrary real t_k *and such that*

(14)
$$\sum_k |a_k| < \infty.$$

Set

$$p(T) = \sum a_k T_{t_k}.$$[20]

If the function $p(\theta)$ *satisfies one or the other of the inequalities*

$$|p(\theta)| \leqq 1, \qquad \text{Re}\, p(\theta) \geqq 0, \text{ for all real } \theta,$$

then

$$\|p(T)\| \leqq 1, \qquad \text{Re}\, p(T) \geqq O$$

respectively.

Proof. The proof proceeds exactly as for c) but now using Theorem IV and Stone's theorem in virtue of which there is a spectral decomposition of $\boldsymbol{U}_t$ of the form

$$\boldsymbol{U}_t = \int_{-\infty}^{\infty} e^{it\lambda}\, d\boldsymbol{E}_\lambda.$$

Analogous theorems could be stated (under suitable hypotheses which assure convergence) for trigonometric integrals.

4. *Isometric* transformations in Hilbert space $\mathfrak{H}$ (into a subspace of $\mathfrak{H}$) are particular cases of contractions. If the isometric transformation T is represented as the projection of a unitary transformation $\boldsymbol{U}$, we have

$$\|f\| = \|Tf\| = \|\boldsymbol{P}\,\boldsymbol{U}f\| \leqq \|\boldsymbol{U}f\|$$

for all f; since, on the other hand, $\|\boldsymbol{U}f\| = \|f\|$, we necessarily have $\boldsymbol{PU}f = \boldsymbol{U}f$, and hence $Tf = \boldsymbol{U}f$; that is, $\boldsymbol{U}$ is an extension of T.

It therefore follows from our theorems on contractions that *every isometric transformation has a unitary extension and that for every weakly continuous one-parameter semi-group of isometric transformations* T_t, *there exists a strongly continuous one-parameter group of unitary transformations* $\boldsymbol{U}_t$ *in an extension space such that* $\boldsymbol{U}_t \supseteq T_t$.

The last theorem was proved earlier by Cooper [3] in an entirely different way.

[20] This series converges in norm because of (14).

5. Normal Extensions

We proved in Sec. 2 in particular that every bounded linear transformation T in the complex Hilbert space $\mathfrak{H}$ can be represented as the projection of a normal transformation in an extension space. The question arises: Does T even have a normal *extension* N?

If a normal extension N of T exists, then a fortiori $T = \operatorname{pr} N$, and consequently $T^* = \operatorname{pr} N^*$, from which it follows that

$$\|Tf\| = \|Nf\| = \|N^*f\| \geqq \|PN^*f\| = \|T^*f\|$$

for all $f \in \mathfrak{H}$. The inequality

$$\|Tf\| \geqq \|T^*f\| \qquad (\text{for all } f \in \mathfrak{H}) \tag{15}$$

is therefore a necessary condition that T have a normal extension. But, it is easy to construct examples of transformations T which do not satisfy this condition.

Other, less simple, necessary conditions are obtained in the following manner. Suppose $\{g_i\}$ $(i = 0, 1, \ldots)$ is a sequence of elements in $\mathfrak{H}$ almost all of which (that is with perhaps the exception of a finite number of them) are equal to the element 0 in $\mathfrak{H}$. We then have

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(T^i g_j, T^j g_i) = \sum_i \sum_j (N^i g_j, N^j g_i) = \sum_i \sum_j (N^{*j} N^i g_j, g_i) =$$
$$= \sum_i \sum_j (N^i N^{*j} g_j, g_i) = \sum_i \sum_j (N^{*j} g_j, N^{*i} g_i) = \|\sum_i N^{*i} g_i\|^2 \geqq 0,$$
$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(T^{i+1} g_j, T^{j+1} g_i) = \|\sum_i (N^*)^{i+1} g_i\|^2 \leqq \|N^*\|^2 \|\sum_i N^{*i} g_i\|^2,$$

from which we see that

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(T^i g_j, T^j g_i) \geqq 0 \tag{16}$$

and

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(T^{i+1} g_j, T^{j+1} g_i) \leqq C^2 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(T^i g_j, T^j g_i) \tag{17}$$

with constant $C > 0$. These two inequalities are therefore *necessary* conditions that T have a bounded normal extension.

But these conditions are also *sufficient*. Namely, the following theorem holds.

THEOREM VI (HALMOS [**1**]). *Every bounded linear transformation T in the Hilbert space $\mathfrak{H}$ which satisfies conditions* (16) *and* (17) *has a bounded normal extension N in an extension space* $\mathbf{H}$. *One can even require that* $\mathbf{H}$ *be minimal in the sense that it is spanned by the elements of the form $N^{*k}f$ where $f \in \mathfrak{H}$ and $k = 0, 1, \ldots$; in this case, the structure $\{\mathbf{H}, N, \mathfrak{H}\}$ is determined to within an isomorphism.*

We shall prove this theorem in Sec. 10 as one of the corollaries to our principal theorem (Sec. 6).

For the present, we shall content ourselves with a remark connecting the problems on extensions with the problems treated in Theorems II-V:

The following three propositions are equivalent for any two bounded linear transformations, T in $\mathfrak{H}$ and $\boldsymbol{T}$ in $\mathbf{H}$ ($\supseteqq \mathfrak{H}$):

a) $T \subseteqq \boldsymbol{T}$;

b) $T = \mathrm{pr}\, \boldsymbol{T}$ *and* $T^*T = \mathrm{pr}\, \boldsymbol{T}^*\boldsymbol{T}$;

c) $T^{*i}T^k = \mathrm{pr}\, \boldsymbol{T}^{*i}\boldsymbol{T}^k$ *for* $i,\ k = 0, 1, \ldots$.

Proof. a) $\to$ c) because

$$(T^{*i}T^k f, g) = (T^k f, T^i g) = (\boldsymbol{T}^k g, \boldsymbol{T}^i g) = (\boldsymbol{T}^{*i}\boldsymbol{T}^k f, g) = (\boldsymbol{P}\boldsymbol{T}^{*i}\boldsymbol{T}^k f, g)$$

for $f, g \in \mathfrak{H}$. c) $\to$ b) is obvious. b) $\to$ a) is proved as follows: for $f \in \mathfrak{H}$ we have, on the one hand, that

$$\|Tf\|^2 = (T^*Tf, f)) = (\boldsymbol{P}\boldsymbol{T}^*\boldsymbol{T}f, f) = (\boldsymbol{T}f, \boldsymbol{T}f) = \|\boldsymbol{T}f\|^2$$

because

$$T^*T = \mathrm{pr}\, \boldsymbol{T}^*\boldsymbol{T},$$

and, on the other hand, that

$$\|Tf\| = \|\boldsymbol{P}\boldsymbol{T}f\|$$

because

$$T = \mathrm{pr}\, \boldsymbol{T}.$$

Hence we have

$$\|\boldsymbol{P}\boldsymbol{T}f\| = \|\boldsymbol{T}f\|$$

which is possible only if $\boldsymbol{T}f = \boldsymbol{P}\boldsymbol{T}f$, that is, if $\boldsymbol{T}f = Tf$; therefore $\boldsymbol{T} \supseteqq T$.

6. Principal Theorem

As we have already stated, Theorems II-VI can be proved as more or less immediate corollaries to a "principal theorem." In order to be able to state this theorem, we must first introduce some concepts of an algebraic nature.

By a *$*$-semi-group* we shall understand a system Γ of elements (which we shall denote by Greek letters) in which two operations are defined: an associative "semi-group operation" $(\xi, \eta) \to \xi\eta$, and a "$*$ operation," $\xi \to \xi^*$, which satisfies the following rules of computation:

$$\xi^{**} = \xi, \quad (\xi\eta)^* = \eta^*\xi^*.$$

We shall assume further that there is a "unit" element ε in Γ such that

$$\varepsilon\xi = \xi\varepsilon = \xi \text{ for all } \xi \in \Gamma, \text{ and } \varepsilon^* = \varepsilon.$$

Any *group* can be considered as a $*$-semi-group if we define the $*$ operation in it as the inverse: $\xi^* = \xi^{-1}$. *In the sequel, when we speak of a group Γ, we shall assume that it is provided with this $*$-semi-group structure.*

By a *representation* of the $*$-semi-group Γ in a Hilbert space $\mathbf{H}$ we shall understand a family $\{\boldsymbol{D}_\xi\}_{\xi\in\Gamma}$ of bounded linear transformations in $\mathbf{H}$ such that

$$D_\varepsilon = I, \quad D_{\xi\eta} = D_\xi D_\eta, \quad D_{\xi^*} = D_\xi^* \quad \text{for all } \xi, \eta \in \Gamma.$$

The following propositions are obvious: if for an $\xi \in \Gamma$.

(i) $\xi^*\xi = \xi\xi^*$, the transformation D_ξ is *normal;*
(ii) $\xi = \xi^*$, the transformation D_ξ is *self-adjoint;*
(iii) $\xi = \xi^* = \xi^2$, the transformation D_ξ is an *(orthogonal) projection;*
(iv) $\xi^*\xi = \xi\xi^* = \varepsilon$, the transformation D_ξ is *unitary.*

Suppose $\{D_\xi\}$ is a representation of Γ in $\mathbf{H}$ and let $\mathfrak{H}$ be a subspace of $\mathbf{H}$. Consider the transformations

$$T_\xi = \mathrm{pr}_{\mathfrak{H}} D_\xi .$$

It is obvious that $T_\varepsilon = I$ (the identity transformation in $\mathfrak{H}$), and that $T_{\xi^*} = \mathrm{pr}\, D_{\xi^*} = \mathrm{pr}\, D_\xi^* = T_\xi^*$. Suppose $\{g_\xi\}_{\xi \in \Gamma}$ is a family of elements in $\mathfrak{H}$ such that $g_\xi = 0$ for almost all the ξ.[21] Upon setting $g = \sum_\xi D_\xi g_\xi$,[22] we then have

$$\sum_\xi \sum_\eta (T_{\xi^*\eta} g_\eta, g_\xi) = \sum_\xi \sum_\eta (D_{\xi^*\eta} g_\eta, g_\xi) = \sum_\xi \sum_\eta (D_\xi^* D_\eta g_\eta, g_\xi) = (g, g) \geqq 0$$

and, for all $\alpha \in \Gamma$,

$$\sum_\xi \sum_\eta (T_{\xi^*\alpha^*\alpha\eta} g_\eta, g_\xi) = \sum_\xi \sum_\eta (D_{\xi^*\alpha^*\alpha\eta} g_\eta, g_\xi) = \sum_\xi \sum_\eta (D_\xi^* D_\alpha^* D_\alpha D_\eta g_\eta, g_\xi) =$$
$$= (D_\alpha g, D_\alpha g) \leqq \|D_\alpha\|^2 (g, g).$$

Now the essential content of our "principal theorem" is that these inequalities characterize the families $\{T_\xi\}$ which are obtained by a projection of a representation $\{D_\xi\}$ of Γ.

PRINCIPAL THEOREM. *Let Γ be a $*$-semi-group and suppose $\{T_\xi\}_{\xi \in \Gamma}$ is a family of bounded linear transformations in the Hilbert space $\mathfrak{H}$ which satisfy the following conditions:*

(a) $T_\varepsilon = I, \quad T_{\xi^*} = T_\xi^*$,[23]

(b) *T_ξ, considered as a function of ξ, is of positive type, that is, for every family $\{g_\xi\}_{\xi \in \Gamma}$ of elements in $\mathfrak{H}$ such that $g_\xi = 0$ for almost all ξ, we have*

$$\sum_\xi \sum_\eta (T_{\xi^*\eta} g_\eta, g_\xi) \geqq 0;$$

(c) *for such families $\{g_\xi\}$ and for all $\alpha \in \Gamma$ we have*

$$\sum_\xi \sum_\eta (T_{\xi^*\alpha^*\alpha\eta} g_\eta, g_\xi) \leqq C_\alpha^2 \sum_\xi \sum_\eta (T_{\xi^*\eta} g_\eta, g_\xi)^{24}$$

with constant $C_\alpha > 0$.

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[21] That is, for all indices with perhaps the exception of a finite number of them. We shall make use of this expression again in the sequel.

[22] The sums extend over all the elements of Γ unless stated expressly otherwise.

[23] In the case of a complex Hilbert space $\mathfrak{H}$ this equation is a consequence of (b).

[24] In particular, the left member is real. This could, after all, have also been deduced from the other hypotheses.

Then there exists a representation $\{\boldsymbol{D}_\xi\}_{\xi\in\Gamma}$ *of* Γ *in an extension space* $\mathbf{H}$ *such that*

$$T_\xi = \mathrm{pr}\, \boldsymbol{D}_\xi .$$

Furthermore, one can require that $\mathbf{H}$ *be minimal in the sense that it is spanned by elements of the form* $\mathbf{D}_\xi f$ *where* $f \in \mathfrak{H}$ *and* $\xi \in \Gamma$. *In this case, the structure* $\{\mathbf{H}, \boldsymbol{D}_\xi, \mathfrak{H}\}_{\xi\in\Gamma}$ *is determined to within an isomorphism and the following propositions are valid:*

1) $\|\boldsymbol{D}_\alpha\| \leqq C_\alpha$;

2) *if the equation* $T_{\xi\alpha\eta} = T_{\xi\beta\eta} + T_{\xi\gamma\eta}$ *is satisfied for fixed* α, β, γ *and for all* $\xi, \eta \in \Gamma$, *we have*

$$\boldsymbol{D}_\alpha = \boldsymbol{D}_\beta + \boldsymbol{D}_\gamma ;$$

3) *if*

$$T_{\xi\alpha_n\eta} \rightharpoonup T_{\xi\alpha\eta} \qquad (n \to \infty)$$

for fixed α_n *and* α, *and for all* ξ, η, *and if furthermore* $\lim C_{\alpha_n} < \infty$, *then we have*

$$\boldsymbol{D}_{\alpha_n} \rightharpoonup \boldsymbol{D}_\alpha \qquad (n \to \infty).$$

Remarks. If Γ is a group, we have $\xi^*\xi = \xi\xi^* = \varepsilon$; hence condition (c) is satisfied in an obvious manner by $C_\alpha = 1$ and the representation $\{\boldsymbol{D}_\xi\}$ consists of unitary transformations.[25]

In the case where Γ is a topological group, and $\mathfrak{H}$ is a complex space of dimension 1, our theorem reduces to a theorem of GELFAND and RAIKOV,[26] in virtue of which every continuous complex-valued function $p(\xi)$ of positive type (see (b) in Principal Theorem, above) defined on Γ can be written in the form

$$p(\xi) = (U_\xi f_0, f_0)$$

where $\{U_\xi\}$ is a weakly (and hence also strongly) continuous[27] unitary representation of Γ in a Hilbert space $\mathfrak{H}$, and where f_0 is a fixed element of $\mathfrak{H}$, an

[25] The principal theorem was already proved for groups in SZ.-NAGY [**11**].

[26] GELFAND-RAIKOV [**1**]. Also see GODEMENT [**2**], in particular pages 21, 22. This theorem is of prime importance in the theory of these authors on irreducible unitary representations of locally bicompact groups.

There is a closely related theorem due to SEGAL [**1**, Theorem 1] on certain complex-valued functions $\omega(A)$, defined on an "operator algebra," that is, on a set $\mathscr{A}$ of bounded linear transformations in a complex Hilbert space, which contains $A + B, AB, cA, A^*$ provided it contains A, B. In the case where $\mathscr{A}$ also contains the transformation I, the SEGAL theorem is also a consequence of our principal theorem.

[27] Since U_ξ is unitary, we have

$$\|U_\xi f - U_\eta f\|^2 = 2\|f\|^2 - 2\,\mathrm{Re}\,(U_\xi f, U_\eta f) = 2\,\mathrm{Re}\,(U_\eta f - U_\xi f, U_\eta f),$$

for $\xi, \eta \in \Gamma$ and $f \in \mathfrak{H}$, from which it follows that the weak continuity of U_ξ implies its strong continuity.

element whose images under the transformations U_ξ span the space $\mathfrak{H}$; under these conditions the structure $\{\mathfrak{H}, U_\xi, f_0\}$ is determined to within an isomorphism.[28]

Proof. 1) *Extension space.* We denote the set of all the families $\boldsymbol{v} = \{v_\xi\}_{\xi\in\Gamma}$ of elements v_ξ in $\mathfrak{H}$ by $\mathbf{V}$; $\boldsymbol{v}$ can be considered also as a vector whose "component with index ξ" is v_ξ, in symbols:

$$(\boldsymbol{v})_\xi = v_\xi .$$

The addition of these vectors and their multiplication by scalars (that is, by real or complex numbers according as $\mathfrak{H}$ is real or complex) are defined by the corresponding operations on components.

In $\mathbf{V}$ we shall consider in particular two linear manifolds, $\mathbf{G}$ and $\mathbf{F}$. $\mathbf{G}$ consists of vectors almost all of whose components are equal to 0; these vectors will be denoted by the letter $\boldsymbol{g}$. $\mathbf{F}$ consists of vectors $\boldsymbol{f} = \{f_\xi\}$[29] for which there exists a vector $\boldsymbol{g} = \{g_\xi\}$ such that

$$f_\xi = \sum_\eta T_{\xi^*\eta} g_\eta$$

for all $\xi \in \Gamma$; this relation between $\boldsymbol{f}$ and $\boldsymbol{g}$ will be denoted by

$$\boldsymbol{f} = \hat{\boldsymbol{g}}.$$

We define a binary form $[\boldsymbol{f},\boldsymbol{f}']$ in $\mathbf{F}$ in the following way. If $\boldsymbol{f} = \hat{\boldsymbol{g}}$, $\boldsymbol{f}' = \hat{\boldsymbol{g}}'$, we let

$$(18) \qquad [\boldsymbol{f},\boldsymbol{f}'] = \sum_\xi (f_\xi, g'_\xi) = \sum_\xi \sum_\eta (T_{\xi^*\eta} g_\eta, g'_\xi) =$$

$$(19) \qquad = \sum_\xi \sum_\eta (g_\eta, T_{\eta^*\xi} g'_\xi) = \sum_\eta (g_\eta, f'_\eta).$$

(Here we have made use of the fact that $T^*_{\xi^*\eta} = T_{(\xi^*\eta)^*} = T_{\eta^*\xi}$.) It follows from (18) that this definition does not depend on the particular choice of $\boldsymbol{g}$ in the representation of $\boldsymbol{f}$, and it follows from (19) that it does not depend on the particular choice of $\boldsymbol{g}'$ either; consequently, the form $[\boldsymbol{f},\boldsymbol{f}']$ is determined uniquely by $\boldsymbol{f}$ and $\boldsymbol{f}'$. It is obviously linear in $\boldsymbol{f}$ and we have $[\boldsymbol{f}',\boldsymbol{f}] = \overline{[\boldsymbol{f},\boldsymbol{f}']}$. It follows from condition (b) that

$$[\boldsymbol{f},\boldsymbol{f}] = \sum_\xi \sum_\eta (T_{\xi^*\eta} g_\eta, g_\xi) \geqq 0.$$

We have still to prove that the equality sign holds here only for $\boldsymbol{f} = 0$. But it follows from what has already been proved that the Schwarz inequality is valid for the form $[\boldsymbol{f},\boldsymbol{f}']$:

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[28] This means that if $\{\mathfrak{H}', U'_\xi, f'_0\}$ is another structure with the same properties, $\mathfrak{H}$ can be mapped linearly and isometrically onto $\mathfrak{H}'$ in such a way that $f_0 \to f'_0$ and that $f \to f'$ implies $U_\xi f \to U'_\xi f'$ for all $\xi \in \Gamma$.

[29] The parameter of the family will be denoted by ξ ; hence ξ always runs through all the elements of Γ.

$$|[f,f']|^2 \leqq [f,f][f',f'].$$

The equation $[f,f]=0$ for one f therefore implies that $[f,f']=0$ for all $f' \in \mathbf{F}$; but it follows easily from (18) that this is possible only if $f=0$.

Hence the form $[f,f']$ possesses all the properties of a scalar product; therefore, if the scalar product in $\mathbf{F}$ is defined by

$$(f,f')=[f,f'],$$

$\mathbf{F}$ becomes a Hilbert space, which in general is *not complete.* Let $\mathbf{H}$ be the *completion* of $\mathbf{F}$.

The original space $\mathfrak{H}$ can be embedded as a subspace in $\mathbf{H}$, and even in $\mathbf{F}$; this can be done by identifying the element f in $\mathfrak{H}$ with the element

$$f_f=\{T_{\xi^*}f\}$$

in $\mathbf{F}$ (note that $f_f=\hat{g}$ with $(g)_\varepsilon=f$ and $(g)_\xi=0$ for $\xi\neq\varepsilon$). This identification is justified because we clearly have

$$f_{cf}=cf_f,\quad f_{f+f'}=f_f+f_{f'},\quad (f_f,f_{f'})=(f,f').$$

Let us now calculate the orthogonal projection $\boldsymbol{P}f$ of an element $f\in\mathbf{F}$ onto the subspace $\mathfrak{H}$! We should have, for all $h\in\mathfrak{H}$,

$$(\boldsymbol{P}f,h)=(f,h),$$

the definition of the scalar product in $\mathbf{F}$ yields

$$(\boldsymbol{P}f,h)=(f,f_h)=((f)_\varepsilon,h);$$

since $\boldsymbol{P}f$ and $(f)_\varepsilon$ are in $\mathfrak{H}$, this equation is possible for all $h\in\mathfrak{H}$ only if

(20) $$\boldsymbol{P}f=(f)_\varepsilon.$$

2) *The representation* $\{\boldsymbol{D}_\xi\}$. Suppose $f=\hat{g}$, that is,

$$f_\xi=\sum_\eta T_{\xi^*\eta}g_\eta.$$

We then have

$$f_{\alpha^*\xi}=\sum_\eta T_{\xi^*\alpha\eta}g_\eta=\sum_\zeta T_{\xi^*\zeta}g_\zeta^\alpha$$

for arbitrary $\alpha\in\Gamma$, where

(21) $$g_\zeta^\alpha=\sum_{\alpha\eta=\zeta}g_\eta$$

(if there are no η such that $\alpha\eta=\zeta$, then the sum in the second member of (21) is defined to be equal to 0). It is clear that, for given α, $g_\xi^\alpha=0$ for almost all the ξ, and therefore

$$\{g_\xi^\alpha\}\in\mathbf{G},\quad \{f_{\alpha^*\xi}\}\in\mathbf{F}.$$

Consequently,

$$\boldsymbol{D}_\alpha\{f_\xi\}=\{f_{\alpha^*\xi}\}$$

is a transformation, which is obviously linear, of $\mathbf{F}$ into $\mathbf{F}$. We have

$$\boldsymbol{D}_\varepsilon\{f_\xi\} = \{f_{\varepsilon^*\xi}\} = \{f_\xi\}, \tag{22}$$

$$\boldsymbol{D}_\alpha \boldsymbol{D}_\beta\{f_\xi\} = \boldsymbol{D}_\alpha\{f_{\beta^*\xi}\} = \{f_{\beta^*\alpha^*\xi}\} = \{f_{(\alpha\beta)^*\xi}\} = \boldsymbol{D}_{\alpha\beta}\{f_\xi\} \tag{23}$$

and, for $\boldsymbol{f} = \hat{\mathbf{g}}$, $\boldsymbol{f}' = \hat{\mathbf{g}}'$,

$$\begin{aligned}(\boldsymbol{D}_\alpha \boldsymbol{f}, \boldsymbol{f}') &= \sum_\xi (f_{\alpha^*\xi}, g'_\xi) = \sum_\xi \sum_\eta (T_{\xi^*\alpha\eta} g_\eta, g'_\xi) = \\ &= \sum_\xi \sum_\eta (g_\eta, T_{\eta^*\alpha^*\xi} g'_\xi) = \sum_\eta (g_\eta, f'_{\alpha\eta}) = (\boldsymbol{f}, \boldsymbol{D}_{\alpha^*}\boldsymbol{f}');\end{aligned} \tag{24}$$

finally, it follows from (24), (23) and condition (c) of the principal theorem that

$$\begin{aligned}(\boldsymbol{D}_\alpha \boldsymbol{f}, \boldsymbol{D}_\alpha \boldsymbol{f}) = (\boldsymbol{D}_{\alpha^*}\boldsymbol{D}_\alpha \boldsymbol{f}, \boldsymbol{f}) = (\boldsymbol{D}_{\alpha^*\alpha}\boldsymbol{f}, \boldsymbol{f}) &= \sum_\xi \sum_\eta (T_{\xi^*\alpha^*\alpha\eta} g_\eta, g_\xi) \leqq \\ &\leqq C_\alpha^2 \sum_\xi \sum_\eta (T_{\xi^*\eta} g_\eta, g_\xi) = C_\alpha^2 (\boldsymbol{f}, \boldsymbol{f}).\end{aligned}$$

Hence $\boldsymbol{D}_\alpha$ is a bounded linear transformation in $\mathbf{F}$, $\|\boldsymbol{D}_\alpha\| \leqq C_\alpha$, and consequently it can be extended by continuity to $\mathbf{H}$. It follows from (22)-(24) that $\{\boldsymbol{D}_\xi\}$ thus extended will be a representation of Γ in $\mathbf{H}$.

Consider in particular an element f which belongs to $\mathfrak{H}$, $\boldsymbol{f} = f$. We then have

$$\boldsymbol{D}_\alpha \boldsymbol{f} = \boldsymbol{D}_\alpha\{T_{\xi^*} f\} = \{T_{\xi^*\alpha} f\}, \tag{25}$$

and hence, by (20),

$$\boldsymbol{P}\boldsymbol{D}_\alpha f = (\boldsymbol{D}_\alpha f)_\varepsilon = T_\alpha f.$$

This proves that

$$T_\alpha = \mathrm{pr}\, \boldsymbol{D}_\alpha.$$

It also follows from (25) that, for $\boldsymbol{f} = \hat{\mathbf{g}} \in \mathbf{F}$,

$$(\boldsymbol{f})_\xi = \sum_\eta T_{\xi^*\eta} g_\eta = \sum_\eta (\boldsymbol{D}_\eta g_\eta)_\xi = (\sum_\eta \boldsymbol{D}_\eta g_\eta)_\xi,$$

whence

$$\boldsymbol{f} = \sum_\eta \boldsymbol{D}_\eta g_\eta.$$

This means that $\mathbf{F}$ consists of finite sums of elements of the form $\boldsymbol{D}_\eta g$ where $g \in \mathfrak{H}$, $\eta \in \Gamma$; then these elements span the space $\mathbf{H}$, and therefore the extension space $\mathbf{H}$ is *minimal*.

The representation $\{\boldsymbol{D}_\xi\}$ of Γ which we have just constructed also satisfies Propositions 2 and 3 of the principal theorem. This follows from the equation

$$(\boldsymbol{D}_\alpha \boldsymbol{f}, \boldsymbol{f}') = \sum_\xi \sum_\eta (T_{\xi^*\alpha\eta} g_\eta, g'_\xi)$$

(see formula (24)), valid for arbitrary $\boldsymbol{f} = \hat{\mathbf{g}}$, $\boldsymbol{f}' = \hat{\mathbf{g}}'$ $(\in \mathbf{F})$, and from the obvious fact that if the relation

$$(\boldsymbol{D}_\alpha \boldsymbol{f}, \boldsymbol{f}') = (\boldsymbol{D}_\beta \boldsymbol{f}, \boldsymbol{f}') + (\boldsymbol{D}_\gamma \boldsymbol{f}, \boldsymbol{f}'),$$

or the relation

$$(D_{\alpha_n} f, f') \to (D_\alpha f, f') \qquad (n \to \infty)$$

is satisfied for $f, f' \in \mathbf{F}$, and if, moreover, in the second case,

$$\overline{\lim} \, \| D_{\alpha_n} \| < \infty,$$

the same relation is satisfied for all the elements f, f' in $\mathbf{H}$.

3) *Isomorphism.* It remains to investigate the problem: To what extent is the structure $\{\mathbf{H}, D_\xi, \mathfrak{H}\}$ determined? To this end, let us consider any two representations of Γ, $\{D'_\xi\}$ in $\mathbf{H}'$ and $\{D''_\xi\}$ in $\mathbf{H}''$, where $\mathbf{H}'$ and $\mathbf{H}''$ are two extension spaces of $\mathfrak{H}$, and let us assume that

$$\operatorname{pr} D'_\xi = T_\xi, \quad \operatorname{pr} D''_\xi = T_\xi.$$

Furthermore, we shall assume that each of these extension spaces is minimal, i.e. that $\mathbf{H}'$ is spanned by the elements $D'_\xi g$ and $\mathbf{H}''$ by the elements $D''_\xi g$, where $g \in \mathfrak{H}$ and $\xi \in \Gamma$.

Let

$$f'_1 = \sum_\xi D'_\xi g_{1\xi}, \quad f'_2 = \sum_\xi D'_\xi g_{2\xi}$$

be two elements in $\mathbf{H}'$ (with $\{g_{1\xi}\}, \{g_{2\xi}\} \in \mathbf{G}$), and let

$$f''_1 = \sum_\xi D''_\xi g_{1\xi}, \quad f''_2 = \sum_\xi D''_\xi g_{2\xi}$$

be elements in $\mathbf{H}''$. We have

$$(f'_1, f'_2) = \sum_\xi \sum_\eta (D'_\eta g_{1\eta}, D'_\xi g_{2\xi}) = \sum_\xi \sum_\eta (D'_{\xi^\bullet\eta} g_{1\eta}, g_{2\xi}) = \sum_\xi \sum_\eta (T_{\xi^\bullet\eta} g_{1\eta}, g_{2\xi}),$$

and in an analogous manner

$$(f''_1, f''_2) = \sum_\xi \sum_\eta (T_{\xi^\bullet\eta} g_{1\eta}, g_{2\xi}),$$

hence

$$(f'_1, f'_2) = (f''_1, f''_2).$$

Consequently, if we assign the elements

$$f' = \sum_\xi D'_\xi g_\xi, \quad f'' = \sum_\xi D''_\xi g_\xi \tag{26}$$

to the same $\{g_\xi\} \in \mathbf{G}$, this correspondence $f' \leftrightarrow f''$ will be linear and isometric, and it can then be extended by continuity to a linear and isometric mapping of all the elements of $\mathbf{H}'$ onto $\mathbf{H}''$.

In particular, by setting $g_\varepsilon = g$ and $g_\xi = 0$ for $\xi \neq \varepsilon$, we see that each element g of the common subspace $\mathfrak{H}$ corresponds to itself. For all $\alpha \in \Gamma$, we have

$$D'_\alpha \sum_\xi D'_\xi g_\xi = \sum_\xi D'_{\alpha\xi} g_\xi = \sum_\zeta D'_\zeta g^\alpha_\zeta \leftrightarrow \sum_\zeta D''_\zeta g^\alpha_\zeta = \sum_\xi D''_{\alpha\xi} g_\xi = D''_\alpha \sum_\xi D''_\xi g_\xi$$

(see (21)); hence $f' \leftrightarrow f''$ implies that $D'_\alpha f' \leftrightarrow D''_\alpha f''$ for all f', f'' in the form (26), and then, in virtue of the continuity of the transformations D'_α, D''_α, for all $f' \in \mathbf{H}'$ and $f'' \in \mathbf{H}''$.

Therefore the structures $\{\mathbf{H}', D'_\xi, \mathfrak{H}\}$ and $\{\mathbf{H}'', D''_\xi, \mathfrak{H}\}$ are *isomorphic*. This completes the proof of the theorem.

7. Proof of the Neumark Theorem

Let $\{B_\lambda\}_{-\infty<\lambda<\infty}$ be a generalized spectral family in $\mathfrak{H}$. Set $B_\infty = \lim_{\lambda\to\infty} B_\lambda = I$ and $B_{-\infty} = \lim_{\lambda\to-\infty} B_\lambda = O$. We assign the transformation

$$B_\Delta = B_b - B_a$$

to each half-open interval

$$\Delta = (a,b] \qquad \text{(where } -\infty \leqq a < b \leqq \infty)$$

and the transformation

$$B_\omega = \sum_i B_{\Delta_i};$$

to each set ω which consists of a finite number of disjoint intervals Δ_i; this definition obviously does not depend on the particular choice of the decomposition of ω. For $\Omega = (-\infty, \infty]$ we have $B_\Omega = I$, and for the void set Θ we have $B_\Theta = O$. The family K of these sets ω, including Ω and Θ, is clearly closed with respect to subtraction of any two sets, and with respect to the operation of forming unions and intersections of a finite number of sets. B_ω is a positive additive set function defined on K; more precisely, B_ω is, for all $\omega \in K$, a self-adjoint transformation such that

$$O \leqq B \leqq I, \quad B_\Theta = O, \quad B_\Omega = I, \quad B_{\omega_1 \cup \omega_2} = B_{\omega_1} + B_{\omega_2} \quad \text{si} \quad \omega_1 \cap \omega_2 = \Theta.$$

We shall also consider K as a $*$-semi-group; we do this by setting

$$\omega_1\omega_2 = \omega_1 \cap \omega_2, \quad \omega^* = \omega, \quad \varepsilon = \Omega.$$

We shall see that B_ω, considered as a function defined on this $*$-semi-group, satisfies the conditions of the principal theorem.

Condition (a) is satisfied in an obvious manner. Condition (b) means that

$$s = \sum_i \sum_j (B_{\omega_i \cap \omega_j} g_j, g_i) \geqq 0$$

for arbitrary $\omega_1, \ldots, \omega_n \in K$ and $g_1, \ldots, g_n \in \mathfrak{H}$. In order to prove this inequality, we first consider the intersections

$$\pi = \omega_1^\pm \cap \omega_2^\pm \cap \cdots \cap \omega_n^\pm \qquad (\in K)$$

where each time we can choose one of the signs $+$ or $-$ in an arbitrary manner; ω^+ denotes the set ω itself and ω^- its complement $\Omega - \omega$. Two intersections π corresponding to different variations of sign are obviously disjoint. Each set $\omega_i \cap \omega_j$ $(i \leqq j)$ is the union of certain of these π, namely of all those obtained by choosing the sign $+$ for i and j, that is, of all those which are contained in $\omega_i \cap \omega_j$. In virtue of the additivity of B_ω as a set function, the sum s then decomposes into a sum of terms of the form

$$(B_\pi g_j, g_i).$$

We combine the terms corresponding to the same π into a partial sum s_π; the latter extends to all the pairs of indices (i, j) for which $\omega_i \cap \omega_j \supseteq \pi$, that is, for

which $\omega_i \supseteqq \pi$ and $\omega_j \supseteqq \pi$ simultaneously. Suppose $i_1, i_2, \ldots, i_r$ are those values of the index i for which ω_i contains the fixed set π; we then have

$$s_\pi = \sum_{h=1}^{r} \sum_{k=1}^{r} (B_\pi g_{i_k}, g_{i_h}) = (B_\pi g, g) \quad \text{with} \quad g = \sum_{h=1}^{r} g_{i_h},$$

and consequently $s_\pi \geqq 0$. Since this is true for all the π, it follows that $s = \sum_\pi s_\pi \geqq 0$, which was to be proved.

Let us now pass on to condition (c). Suppose ω is a fixed element in K and set

$$\omega_i' = \omega_i \cap \omega^+, \quad \omega_i'' = \omega_i \cap \omega^- \qquad (i = 1, 2, \ldots, n).$$

Applying the inequality $s \geqq 0$, which we have just proved, to ω_i' and ω_i'' instead of to ω_i, we obtain

$$s' = \sum_i \sum_j (B_{\omega_i' \cap \omega_j'} g_j, g_i) \geqq 0, \quad s'' = \sum_i \sum_j (B_{\omega_i'' \cap \omega_j''} g_j, g_i) \geqq 0.$$

Since $\omega_i' \cap \omega_j'$ and $\omega_i'' \cap \omega_j''$ are to be contained in the disjoint sets ω^+, ω^-, they are also disjoint; since their union is equal to $\omega_i \cap \omega_j$, it follows from the additivity of B_ω that $s' + s'' = s$. Consequently, we have $0 \leqq s' \leqq s$, that is

$$0 \leqq \sum_i \sum_j (B_{\omega_i \cap \omega \cap \omega \cap \omega_j} g_j, g_i) \leqq \sum_i \sum_j B_{\omega_i \cap \omega_j} g_j, g_i),$$

and hence condition (c) is satisfied with $C_\omega = 1$.

We can then apply the principal theorem. Hence there exists a representation $\{\boldsymbol{E}_\omega\}$ of the $*$-semi-group K in a minimal extension space $\mathbf{H}$ such that

$$B_\omega = \operatorname{pr} \boldsymbol{E}_\omega;$$

here, "minimal" means that the space $\mathbf{H}$ is spanned by the elements of the form $\boldsymbol{E}_\omega f$ where $f \in \mathfrak{H}$, $\omega \in K$. It follows from the structure of K as a $*$-semi-group that $\boldsymbol{E}_\omega$ is a projection, $\boldsymbol{E}_\Omega = \boldsymbol{I}$, and

(27) $$\boldsymbol{E}_{\omega_1 \cap \omega_2} = \boldsymbol{E}_{\omega_1} \boldsymbol{E}_{\omega_2}.$$

We also have

(28) $$\boldsymbol{E}_{\omega_1 \cup \omega_2} = \boldsymbol{E}_{\omega_1} + \boldsymbol{E}_{\omega_2} \quad \text{when} \quad \omega_1 \cap \omega_2 = \Theta;$$

this follows, in virtue of the fact that $\mathbf{H}$ is minimal, from the fact that, B_ω being an additive function of ω, we have

$$B_{(\omega_1 \cup \omega_2) \cap \omega} = B_{\omega_1 \cap \omega} + B_{\omega_2 \cap \omega}$$

for all $\omega \in K$. In particular, we have $\boldsymbol{E}_\Theta = \boldsymbol{E}_{\Theta \cup \Theta} = \boldsymbol{E}_\Theta + \boldsymbol{E}_\Theta$, and hence $\boldsymbol{E}_\Theta = \boldsymbol{O}$. We set

$$\boldsymbol{E}_\lambda = \boldsymbol{E}_{(-\infty, \lambda]} \text{ for } -\infty < \lambda < \infty;$$

since $B_{(-\infty, \lambda]} = B_\lambda - B_{-\infty} = B_\lambda$, we then have

(29) $$B_\lambda = \operatorname{pr} \boldsymbol{E}_\lambda,$$

and by (27) or (28),

$$\boldsymbol{E}_{\mu} \leqq \boldsymbol{E}_{\mu} \text{ for } \lambda < \mu.$$

We finally arrive at the relations

$$\begin{aligned} &\boldsymbol{E}_{\lambda} \to \boldsymbol{E}_{\mu} && \text{as } \lambda \to \mu + 0; \\ &\boldsymbol{E}_{\lambda} \to \boldsymbol{E}_{\Theta} = \boldsymbol{O} && \text{as } \lambda \to -\infty; \\ &\boldsymbol{E}_{\lambda} \to \boldsymbol{E}_{\Omega} = \boldsymbol{I} && \text{as } \lambda \to \infty,^{30} \end{aligned}$$

which are consequences, in virtue of the fact that $\mathbf{H}$ is minimal, of the relations

$$\begin{aligned} &B_{(-\infty,\lambda]\cap\omega} \to B_{(-\infty,\mu]\cap\omega} && \text{as } \lambda \to \mu + 0, \\ &B_{(-\infty,\lambda]\cap\omega} \to O = B_{\Theta\cap\omega} && \text{as } \lambda \to -\infty, \\ &B_{(-\infty,\lambda]\cap\omega} \to B_{\omega} = B_{\Omega\cap\omega} && \text{as } \lambda \to +\infty, \end{aligned}$$

which are valid for all fixed ω.

Hence $\{\boldsymbol{E}_{\omega}\}$ is an ordinary spectral family.

Since each of the $\boldsymbol{E}_{\omega}$ is derived from the $\boldsymbol{E}_{\lambda}$ by forming differences and sums or by passing to the limit ($\lambda \to \pm\infty$), the space $\mathbf{H}$ is also minimal with respect to $\{\boldsymbol{E}_{\lambda}\}$, and the structure $\{\mathbf{H}, \boldsymbol{E}_{\lambda}, \mathfrak{H}\}$ is determined to within an isomorphism.

This completes the proof of Theorem I. The following theorem, also due to NEUMARK [5], is proved in an analogous manner.

THEOREM. *Suppose K is a family of subsets ω of a set Ω which contains Ω and the void set Θ, and which is closed with respect to subtraction of sets, as well as with respect to forming the union and intersection of a finite number of sets. We assign to each $\omega \in K$ a self-adjoint transformation B_{ω} in the Hilbert space $\mathfrak{H}$, in such a way that we have*

$$O \leqq B_{\omega} \leqq I; \quad B_{\Theta} = O; \quad B_{\Omega} = I; \quad B_{\omega_1 \cup \omega_2} = B_{\omega_1} + B_{\omega_2} \quad si \quad \omega_1 \cap \omega_2 = \Theta.$$

Then there exists a family $\{\mathbf{E}_{\omega}\}$ of projections in an extension space $\mathbf{H}$, such that the elements of the form $\mathbf{E}_{\omega} f$ ($f \in \mathfrak{H}, \omega \in K$) determine the space $\mathbf{H}$, and

$$B_{\omega} = \operatorname{pr} \boldsymbol{E}_{\omega};$$

$$\boldsymbol{E}_{\Theta} = \boldsymbol{O}; \quad E_{\Omega} = I;$$

$$\mathbf{E}_{\omega_1 \cap \omega_2} = \mathbf{E}_{\omega_1} \mathbf{E}_{\omega_2} \quad \textit{for arbitrary } \omega_1, \omega_2;$$

$$\mathbf{E}_{\omega_1 \cup \omega_2} = \mathbf{E}_{\omega_1} + \mathbf{E}_{\omega_2} \quad \textit{for disjoint } \omega_1, \omega_2.$$

8. Proof of the Theorem on Sequences of Moments

We saw in Sec. 3 how Theorem II is derivable from Theorem I of Neumark (at least with the exception of the last propositions of Theorem II, which however can be proved directly without difficulty). We are now going to see how this theorem is derivable directly from our principal theorem.

Suppose Γ is the $*$-semi-group of non-negative integers n with addition as the "semi-group operation" and with the identity operation $n^* = n$ as the "$*$ operation"; then the "unit" element is the number 0.

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[30] For a monotone sequence of self-adjoint transformations, the weak limit is at the same time the strong limit, which fact follows from Sec. 104.

Every representation of Γ is obviously of the form $\{A^n\}$ where A is a bounded self-adjoint transformation.

We shall show that the sequence $\{A_n\}$ $(n=0,1,\ldots)$ visualized in Theorem II, considered as a function of the variable element n in the $*$-semi-group Γ, satisfies the conditions of the principal theorem. Condition (a) is obviously satisfied; as for the other two conditions, one proves them using the integral formula

$$A_n = \int_{-M-0}^{M} \lambda^n \, dB_\lambda$$

established in Sec. 3, where $\{B_\lambda\}$ is a generalized spectral family on the interval $[-M, M]$. In fact, if $\{g_n\}$ $(n=0, 1, \ldots)$ is any sequence of elements in $\mathfrak{H}$, which are almost all equal to 0, we have

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(A_{i+k}g_k, g_i) = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\int_{-M-0}^{M}\lambda^{i+k}\, d(B_\lambda g_k, g_i) = \int_{-M-0}^{M}(B(d\lambda)g(\lambda), g(\lambda)) \geqq 0$$

where we have set

$$g(\lambda) = \sum_{i=0}^{\infty}\lambda^i g_i$$

and where $B(\Delta)$ denotes the positive, additive interval function generated by B_λ, that is, $B(\Delta) = B_b - B_a$ for $\Delta = (a, b]$. Furthermore, for $r = 0, 1, \ldots,$ we have that

$$\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(A_{i+2r+k}g_k, g_i) = \int_{-M-0}^{M}\lambda^{2r}(B(d\lambda)g(\lambda), g(\lambda)) \leqq$$

$$\leqq M^{2r}\int_{-M-0}^{M}(B(d\lambda)g(\lambda), g(\lambda)) = M^{2r}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}(A_{i+k}g_k, g_i).$$

Thus we see that conditions (b) and (c) are also satisfied, and one can then apply the principal theorem, which yields Theorem II.

9. Proof of the Theorems on Contractions

Proof of Theorem III. Suppose Γ is the additive group of all integers n. Every representation of Γ is then of the form $\{U^n\}$ where U is a unitary transformation.

Suppose T is a contraction in $\mathfrak{H}$. Set

$$T_n = \begin{cases} T^n \text{ for } n = 0, 1, \ldots, \\ T^{*|n|} \text{ for } n = -1, -2, \ldots; \end{cases}$$

hence $T_0 = I$ and $T_{-n} = T_n^*$. We shall show that T_n, considered as a function defined on Γ, is of positive type, that is,

$$\sum_m \sum_n (T_{n-m}g_n, g_m) \geqq 0 \tag{31}$$

for every sequence $\{g_n\}_{-\infty}^{\infty}$ of elements in $\mathfrak{H}$ almost all of which are equal to 0.

We first consider the case of a *complex* space $\mathfrak{H}$. We set

$$T(r, \varphi) = \sum_{-\infty}^{\infty} r^n e^{in\varphi} T_n \tag{32}$$

for $0 \leqq r < 1$ and $0 \leqq \varphi \leqq 2\pi$; in view of the fact that $\|T_n\| \leqq 1$, this series converges in norm. Setting $z = re^{i\varphi}$ we have

$$T(r,\varphi) = \left(\frac{1}{2} I + \sum_1^\infty z^n T^n\right) + \left(\frac{1}{2} I + \sum_1^\infty \bar{z}^n T^{*n}\right) =$$

$$= 2 \operatorname{Re}\left(\frac{1}{2} I + \sum_1^\infty z^n T^n\right) = \operatorname{Re}(I + zT)(I - zT)^{-1}.$$

Hence, for $f \in \mathfrak{H}$ and $g = (I - zT)^{-1} f$, we have

$$\begin{aligned}(33)\quad &(T(r,\varphi)f, f) = \operatorname{Re}((I+zT)(I-zT)^{-1}f, f) = \operatorname{Re}((I+zT)g, (I-zT)g) = \\ &= \operatorname{Re}[(g,g) + z(Tg,g) - \bar{z}(g,Tg) - z\bar{z}(Tg,Tg)] = \|g\|^2 - |z|^2\|Tg\|^2 \geqq 0,\end{aligned}$$

since $|z| < 1$, $\|T\| \leqq 1$. Since this result holds for all $f \in \mathfrak{H}$, we have in particular that

$$(34)\qquad p(r,\varphi) = (T(r,\varphi)f(\varphi), f(\varphi)) \geqq 0$$

with

$$f(\varphi) = \sum_{n=-\infty}^{\infty} e^{-in\varphi} g_n$$

where $\{g_n\}$ is the sequence of elements in $\mathfrak{H}$ considered in inequality (31). If in (34) we replace $T(r,\varphi)$ and $f(\varphi)$ by their series expansions, we obtain that

$$p(r,\varphi) = \sum_{k,m,n} r^{|k|} e^{i(k+m-n)\varphi} (T_k g_n, g_m) =$$

$$= \sum_l e^{il\varphi} \sum_{m,n} r^{|l+n-m|} (T_{l+n-m} g_n, g_m) \geqq 0,$$

whence

$$\sum_{m,n} r^{|n-m|} (T_{n-m} g_n, g_m) = \frac{1}{2\pi} \int_0^{2\pi} p(r,\varphi)\, d\varphi \geqq 0.$$

We obtain (31) by letting $r \to 1$.

The case of a *real* space $\mathfrak{H}$ can be reduced to that of a complex space; we do this by introducing the space $\mathfrak{H}_c$ of pairs $\{g, h\}$ of elements in $\mathfrak{H}$, subject to the following fundamental operations:

$$\{g,h\} + \{g',h'\} = \{g+g', h+h'\},$$

$$(a+ib)\{g,h\} = \{ag - bh,\ bg + ah\} \quad (a \text{ and } b \text{ are real numbers}),$$

$$(\{g,h\}, \{g',h'\}) = (g,g') + (h,h') + i(h,g') - i(g,h'),$$

$$\|\{g,h\}\|^2 = (\{g,h\},\{g,h\}) = \|g\|^2 + \|h\|^2;$$

hence $\mathfrak{H}_c$ is a complex Hilbert space. The transformation

$$\bar{T}\{g,h\} = \{Tg, Th\}$$

is a contraction in $\mathfrak{H}_c$; in fact, on the one hand, it is linear:

$$\bar{T}\{g+g', h+h'\} = \{T(g+g'), T(h+h')\} =$$

$$= \{Tg, Th\} + \{Tg', Th'\} = \bar{T}\{g,h\} + \bar{T}\{g',h'\},$$

$$\bar{T}(a+ib)\{g,h\} = \bar{T}\{ag - bh, bg + ah\} = \{aTg - bTh, bTg + aTh\} =$$

$$= (a+ib)\{Tg, Th\} = (a+ib)\ \bar{T}\{g,h\},$$

and, on the other hand, we have

$$\|\bar{T}\{g,h\}\|^2=\|\{Tg,Th\}\|^2=\|Tg\|^2+\|Th\|^2\leqq\|g\|^2+\|h\|^2=\|\{g,h\}\|^2.$$

It is also easily seen that

$$\bar{T}^*\{g,h\}=\{T^*g,T^*h\}.$$

It follows that

$$\bar{T}^n\{g,h\}=\{T^ng,T^nh\} \text{ and } \bar{T}^{*n}\{g,h\}=\{T^{*n}g,T^{*n}h\}$$

for $n=0, 1, \ldots$, and hence

$$\bar{T}_n\{g,h\}=\{T_ng,T_nh\}$$

for $n=0,\pm 1,\pm 2,\ldots.$

But since inequality (31) has already been proved for complex spaces, we shall have

$$\sum_m\sum_n(\bar{T}_{n-m}\varphi_n,\varphi_m)\geqq 0 \tag{35}$$

for $\varphi_n=\{g_n,h_n\}$ (where $g_n=0$ and $h_n=0$ for almost all n). When $h_n=0$ for all n, we have

$$(\bar{T}_{n-m}\varphi_n,\varphi_m)=(T_{n-m}g_n,g_m),$$

and hence inequality (35) then reduces to inequality (31), which completes the proof of (31) also in the case of a real space $\mathfrak{H}$.

We can then apply the principal theorem, which yields Theorem III.

Proof of Theorem IV. Now let Γ be the additive group of all real numbers t. Then the representations of Γ are one-parameter groups $\{U_t\}$ of unitary transformations.

Let $\{T_t\}_{t\geqq 0}$ be the one-parameter semi-group of contractions considered in the theorem. We set

$$T_t=T^*_{-t}$$

for $t<0$; then T_t will be a weakly continuous function of t, $-\infty<t<\infty$, and we shall have

$$T_0=I \text{ and } T_{-t}=T_t^* \text{ for } -\infty<t<\infty.$$

We shall show that T_t, considered as a function on Γ, is of positive type, that is, that

$$\sum_s\sum_t(T_{t-s}h_t,h_s)\geqq 0 \tag{36}$$

for every family $\{h_t\}$ of elements in $\mathfrak{H}$ such that $h_t=0$ for almost all values of t.

Suppose $t_1, t_2, \ldots t_r$ are those values of t for which $h_t\neq 0$. We assign to each $t_n (n=1,\ldots,r)$ a sequence of rational numbers $t_{n\nu}$ $(\nu=1,2,\ldots)$ which converges to t in such a manner that the numbers $t_{n\nu}$ $(n=1,2,\ldots,r)$ are distinct for every fixed index ν. Since T_t is a weakly continuous function of t, setting

$$f_n = h_{t_n} \qquad (n = 1, 2, \ldots, r),$$

we have

(37)
$$\sum_s \sum_t (T_{t-s} h_t, h_s) = \sum_{m=1}^{r} \sum_{n=1}^{r} (T_{t_n - t_m} f_n, f_m) =$$
$$= \lim_{\nu \to \infty} \sum_{m=1}^{r} \sum_{n=1}^{r} (T_{t_{n\nu} - t_{m\nu}} f_n, f_m).$$

For every fixed ν, the rational numbers $t_{n\nu}$ $(n = 1, \ldots, r)$ are commensurable, that is, they can be written in the form

$$t_{n\nu} = \tau_{n\nu} d_\nu$$

with a $d_\nu > 0$ and distinct integers $\tau_{n\nu}$. Then we have

$$T_{t_{n\nu} - t_{m\nu}} = T_{(\tau_{n\nu} - \tau_{m\nu}) d_\nu} = \begin{cases} (T_{d_\nu})^{\tau_{n\nu} - \tau_{m\nu}} & \text{when } \tau_{n\nu} \geqq \tau_{m\nu}, \\ (T^*_{d_\nu})^{\tau_{m\nu} - \tau_{n\nu}} & \text{when } \tau_{n\nu} \leqq \tau_{m\nu}, \end{cases}$$

and hence

(38)
$$\sum_{m=1}^{r} \sum_{n=1}^{r} (T_{t_{n\nu} - t_{m\nu}} f_n, f_m) = \sum_{m=1}^{r} \sum_{n=1}^{r} (T^{(\nu)}_{\tau_{n\nu} - \tau_{m\nu}} f_n, f_m)$$

where $T^{(\nu)}_n$ is defined in a manner analogous to (30), starting with the transformation $T^{(\nu)} = T_{d_\nu}$. Since the latter is a contraction, inequality (31) holds for it also; choosing the g_n in (31) in such a way that

$$g_n = f_p \text{ when } n = \tau_{p\nu},$$
$$g_n = 0 \text{ when } n \text{ is not equal to any of the } \tau_{q\nu} \ (q = 1, 2, \ldots, r),$$

the first member of inequality (31) reduces to the second member of equation (38), and hence the latter is $\geqq 0$; and this is true for all fixed values of ν. Inequality (36) follows, in virtue of (37).

Then we can apply the principal theorem and obtain that

$$T_t = \operatorname{pr} \boldsymbol{U}_t,$$

and that in the case where the extension space $\mathbf{H}$ in question is minimal, the structure $\{\mathbf{H}, \boldsymbol{U}_t, \mathfrak{H}\}$ is determined to within an isomorphism. In this case, $\boldsymbol{U}_t$ is also a weakly (and hence strongly) continuous function of t, and this in virtue of proposition 3) of the principal theorem and because of the fact that T_{t+t_0} is obviously a weakly continuous function of t for an arbitrary fixed value t_0 of t.

This completes the proof of Theorem IV.

Proof of Theorem V. We now choose Γ to be the group of all the "vectors" $\boldsymbol{n} = \{n^{(\rho)}\}_{\rho \in R}$ whose components are integers, almost all of which are equal to 0. If $\{T^{(\rho)}\}_{\rho \in R}$ is the given system of pairwise doubly permutable contractions, we set

(39)
$$T_{\boldsymbol{n}} = \prod_{\rho \in R} T^{(\rho)}_{n^{(\rho)}}$$

where $T^{(\rho)}_n$ is defined in a manner analogous to (30). Since $n^{(\rho)} = 0$ for almost

all ρ, almost all the factors in the product (39) are equal to I; therefore, this product has meaning even in the case where the set R is infinite. We note, which is essential, that since the $T^{(\rho)}$ are pairwise doubly permutable the factors in (39) are all permutable.

We obviously have $T_{\boldsymbol{o}} = I$, $T_{-\boldsymbol{n}} = T^*_{\boldsymbol{n}}$, where $\mathbf{o}$ denotes the vector all of whose components equal zero.

It remains to prove that $T_{\boldsymbol{n}}$, considered as a function on the group Γ, is of positive type, that is

$$\sum_{\boldsymbol{m}} \sum_{\boldsymbol{n}} (T_{\boldsymbol{n}-\boldsymbol{m}} g_{\boldsymbol{n}}, g_{\boldsymbol{m}}) \geqq 0 \tag{40}$$

for every family $\{g_{\boldsymbol{n}}\}$ of elements in $\mathfrak{H}$ such that $g_{\boldsymbol{n}} = 0$ for almost all $\boldsymbol{n} \in \Gamma$.

If one considers only those vectors $\boldsymbol{n}$ for which $g_{\boldsymbol{n}} \neq 0$, there is a finite number of indices ρ, say $\rho_1, \rho_2, \ldots, \rho_r$, such that all the components of the vectors $\boldsymbol{n}$ whose indices are different from these, are equal to 0. Since the factors with $n^{(\rho)} = 0$ in the product (39) can be omitted, it suffices to consider the sums of the type

$$\sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_r=-\infty}^{\infty} (T^{(1)}_{n_1-m_1} \cdots T^{(r)}_{n_r-m_r} g_{n_1,\ldots,n_r}, g_{m_1,\ldots,m_r}) \tag{41}$$

where we have set $T^{(i)}$ in place of $T^{(\rho_i)}$ for simplicity in writing.

In the case of a complex space $\mathfrak{H}$ one can reason as following. We set

$$T(r, \varphi_1, \ldots, \varphi_r) = \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_r=-\infty}^{\infty} r^{|n_1|+\ldots+|n_r|} e^{i(n_1\varphi_1+\ldots+n_r\varphi_r)} T^{(1)}_{n_1} \cdots T^{(r)}_{n_r} =$$

$$= \prod_{i=1}^{r} T^{(i)}(r, \varphi_i),$$

for $0 \leqq r < 1$ and $0 \leqq \varphi_i \leqq 2\pi$, where the factors in the last member have a meaning analogous to (32). Since these factors are, according to (33), $\geqq O$, and since they are pairwise permutable, their product is also $\geqq O$. Hence, we have in particular that

$$(T(r, \varphi_1, \ldots, \varphi_r) g(\varphi_1, \ldots, \varphi_r), g(\varphi_1, \ldots, \varphi_r)) \geqq 0$$

with

$$g(\varphi_1, \ldots, \varphi_r) = \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_r=-\infty}^{\infty} e^{-i(n_1\varphi_1+\ldots+n_r\varphi_r)} g_{n_1,\ldots,n_r}.$$

Integrating with respect to each variable φ_i from 0 to 2π, and then letting r tend to 1, we have the result that the sum (41) is $\geqq 0$.

This proves inequality (40) for the case of a complex space. The case of a real space can be reduced to that of a complex space in the same way this was done in the proof of Theorem III.

The principal theorem can then be applied. In order to obtain Theorem V, it only remains to observe that every representation $\{U_{\boldsymbol{n}}\}$ of the group Γ is of the form

$$U_{\boldsymbol{n}} = \prod_{\varrho \in R} [U^{(\varrho)}]^{n^{(\varrho)}} \qquad (\boldsymbol{n} = \{n^{(\varrho)}\})$$

where $\{U^{(\rho)}\}$ is a system of permutable unitary transformations. This follows from the fact that n can be written in the form

$$n = \sum_{\varrho \in R} n^{(\varrho)} e_\varrho$$

where e_ρ denotes the vector all of whose components equal zero except the component with index ρ, which is equal to 1; all one has to do is set

$$U^{(\varrho)} = U_{e_\varrho}.$$

10. Proof of the Theorem on Normal Extensions

Now let Γ be the following $*$-semi-group: its elements are the ordered pairs $\pi = \{i, j\}$ of non-negative integers; the semi-group operation is defined in it by

$$\pi + \pi' = \{i, j\} + \{i', j'\} = \{i + i', j + j'\}$$

and the $*$ operation by

$$\pi^* = \{i, j\}^* = \{j, i\};$$

then the "unit" element is

$$\varepsilon = \{0, 0\}.$$

Let $\{D_\pi\}$ be a representation of Γ. Since the semi-group operation in Γ is commutative, the transformations D_π are *normal* and pairwise doubly permutable. If we set $\eta = \{0, 1\}$, every $\pi = \{i, j\}$ can be written in the form

$$\pi = i\eta^* + j\eta,$$

and consequently we have

$$D_\pi = N^{*i} N^j$$

where

$$N = D_\eta.$$

Let T be a bounded linear transformation in the space $\mathfrak{H}$ which satisfies the conditions of Theorem VI:

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (T^i g_j, T^j g_i) \geqq 0, \tag{42}$$

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (T^{i+1} g_j, T^{j+1} g_i) \leqq C^2 \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (T^i g_j, T^j g_i) \qquad (C > 0). \tag{43}$$

We shall show that the transformation

$$T_{\{i, j\}} = T^{*i} T^j,$$

considered as a function defined on the $*$-semi-group Γ, satisfies the conditions of the principal theorem.

It is first of all obvious that $T_\varepsilon = I$, $T_{\pi^*} = T_\pi^*$. In order to prove that T_π is a function of positive type, we choose any family $\{g_\pi\}$ of elements in $\mathfrak{H}$, almost all of which are equal to 0, and consider the sum

$$s = \sum_\pi \sum_{\pi'} (T_{\pi^* + \pi'} g_{\pi'}, g_\pi)$$

where $\pi = \{i,j\}$ and $\pi' = \{i',j'\}$ run through the elements of Γ. It is easy to see that

$$s = \sum_\pi \sum_{\pi'} ((T^*)^{i'+j} T^{i+j'} g_{\pi'}, g_\pi) = \sum_i \sum_{i'} (T^i h_{i'}, T^{i'} h_i)$$

where

$$h_i = \sum_j T^j g_{\{i,j\}}.$$

In virtue of (42), it follows from this that $s \geqq 0$ and therefore T_π is a function of positive type.

Repeating inequality (43) $(i_0 + j_0)$ times we obtain, in an analogous manner, that for fixed $\pi_0 = \{i_0, j_0\}$, we have

$$\sum_\pi \sum_{\pi'} (T_{\pi^* + \pi_0^* + \pi_0 + \pi'} g_{\pi'}, g_\pi) = \sum_i \sum_{i'} (T^{i_0+j_0+i} h_{i'}, T^{i_0+j_0+i'} h_i) \leqq$$

$$\leqq C^{2(i_0+j_0)} \sum_i \sum_{i'} (T^i h_{i'}, T^{i'} h_i) = C^{2(i_0+j_0)} \sum_\pi \sum_{\pi'} (T_{\pi^* + \pi'} g_{\pi'}, g_\pi).$$

Conditions (a) - (c) of the principal theorem are therefore satisfied, and then applying this theorem, it follows that, in an extension space $\mathbf{H}$, there exists a bounded normal transformation $\boldsymbol{N}$ (with $\|\boldsymbol{N}\| \leqq C$) such that

$$T^{*i} T^j = \mathrm{pr}\, \boldsymbol{N}^{*i} \boldsymbol{N}^j \qquad (i, j = 0, 1, \ldots),$$

which is equivalent to $T \subseteqq \boldsymbol{N}$ (see Sec. 5). If $\mathbf{H}$ is minimal in the sense that it is spanned by the elements $\boldsymbol{N}^{*i} \boldsymbol{N}^j f$ $(f \in \mathfrak{H})$, it is also spanned by the elements $\boldsymbol{N}^{*i} f$, inasmuch as $\boldsymbol{N}^j f \in \mathfrak{H}$ for $f \in \mathfrak{H}$.

This completes the proof of Theorem VI.

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